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Representations of Lie superalgebras in prime characteristic II: The queer series

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ABSTRACT

The modular representation theory of the queer Lie superalgebra $\mathfrak{q}(n)$ over characteristic $p > 2$ is developed. We obtain a criterion for the irreducibility of baby Verma modules with semisimple p-characters χ and a criterion for the semisimplicity of the corresponding reduced enveloping algebras $U_\chi(\mathfrak{q}(n))$. A $(2p)$-power divisibility of dimensions of $\mathfrak{q}(n)$-modules with nilpotent p-characters is established. The representation theory of $\mathfrak{q}(2)$ is treated in detail. We formulate a Morita super-equivalence conjecture for $\mathfrak{q}(n)$ with general p-characters which is verified for $n = 2$.

1. Introduction

1.1. In [18], the authors initiated the modular representation theory of Lie superalgebras over an algebraically closed field K of characteristic $p > 2$, by formulating a general superalgebra analogue of the Kac–Weisfeiler (KW) conjecture and establishing it for the basic classical Lie superalgebras. Our work generalized (via a somewhat modified approach) the earlier work on Lie algebras of reductive algebraic groups by Kac and Weisfeiler [19], Parshall and Friedlander [5], Premet [13,14], and others (cf. Jantzen [11] for a review and extensive references).

This paper is a sequel to [18], and its goal is to develop systematically the modular representation theory of the queer Lie superalgebra $\mathfrak{g} \equiv \mathfrak{q}(n)$ over the field K. As a byproduct, the finite W-algebra associated with $\mathfrak{g}$ is also introduced.

1.2. Recall that the queer Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ consists of matrices of the form

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}, \tag{1.1}$$

where A and B are arbitrary $n \times n$ matrices. Note that the even subalgebra $\mathfrak{g}_{\bar{0}}$ is isomorphic to $\mathfrak{gl}(n)$ and the odd part $\mathfrak{g}_{\bar{1}}$ is another isomorphic copy of $\mathfrak{gl}(n)$ under the adjoint action of $\mathfrak{g}_{\bar{0}}$. The queer Lie superalgebra $\mathfrak{g}$ can be regarded as a true super-analogue of the general linear Lie algebra, and its representation theory over the complex field has been studied by various authors (see [2,4,12,16,6]). Also the modular representations of the type Q algebraic supergroup have been studied in [3] and they played a key role in the classification of the simple modules of the spin symmetric group over K.

It is worth emphasizing that, in contrast to simple Lie algebras, a Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_{\bar{0}} + \mathfrak{h}_{\bar{1}}$ of the queer Lie superalgebra is not abelian and its odd part $\mathfrak{h}_{\bar{1}}$ is non-zero. Moreover, $\mathfrak{q}(n)$ admits a non-degenerate *odd* symmetric bilinear form.

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For restricted Lie superalgebras including $\mathfrak{g}$, one can make sense of the notions of p-characters $\chi \in \mathfrak{g}_{\bar{0}}^*$ and the corresponding reduced enveloping algebras $U_\chi(\mathfrak{g})$ (see Section 2). Recalling $\mathfrak{g}_{\bar{0}} \cong \mathfrak{gl}(n)$, one can also make sense of the Jordan decomposition of a p-character as well as the notion of semisimple and nilpotent p-characters. The Lie superalgebra $\mathfrak{g}$ admits a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. We may assume that a p-character χ satisfies $\chi(\mathfrak{n}_{\bar{0}}^+) = 0$ without loss of generality, via a $GL(n)$-conjugation if necessary. For a weight λ in a certain subset Λ_χ of $\mathfrak{h}_{\bar{0}}^*$ (see (2.1)), we define the simple $U_\chi(\mathfrak{h})$-module $V_\chi(\lambda)$ (which is in general not one-dimensional because $\mathfrak{h}$ is non-abelian), and then define the baby Verma module $Z_\chi(\lambda) = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h}\oplus\mathfrak{n}^+)} V_\chi(\lambda)$. Note that baby Verma modules have varied dimensions depending on λ.

1.3. Our first main result is the following criterion on the irreducibility of $Z_\chi(\lambda)$ (see (3.2) for the precise definition of the polynomial Φ).

Theorem A (*Theorem 3.4*)**.** *Assume that $\chi \in \mathfrak{g}_{\bar{0}}^*$ is semisimple with $\chi(\mathfrak{n}_{\bar{0}}^+) = \chi(\mathfrak{n}_{\bar{0}}^-) = 0$. Then a baby Verma module $Z_\chi(\lambda)$ with $\lambda \in \Lambda_\chi$ is irreducible if and only if $\Phi(\lambda) \neq 0$.*

This should be regarded as a queer analogue of Rudakov's classical result for modular Lie algebras [15]. We need some extra care in dealing with the complication from the multi-dimensionality of the high weight subspace $V_\chi(\lambda)$ of $Z_\chi(\lambda)$. An immediate consequence of the above theorem is a neat criterion for the semisimplicity of $U_\chi(\mathfrak{g})$ associated with semisimple p-characters χ. Denote by $h_i \in \mathfrak{h}_{\bar{0}}$ $(1 \leq i \leq n)$ the element corresponding to the ith diagonal matrix unit of A in (1.1).

Theorem B (*Theorem 3.10*)**.** *Let χ be semisimple with $\chi(\mathfrak{n}_{\bar{0}}^+) = \chi(\mathfrak{n}_{\bar{0}}^-) = 0$. The algebra $U_\chi(\mathfrak{g})$ is semisimple if and only if $0 \neq \chi(h_i) \neq \pm\chi(h_j)$ for $1 \leq i \neq j \leq n$.*

1.4. Another main result of the paper is the proof of the super-KW conjecture, which was first formulated in [18], for $\mathfrak{g} = \mathfrak{q}(n)$ with nilpotent p-characters. Let $\chi \in \mathfrak{g}_{\bar{0}}^*$ be nilpotent. We consider $\chi \in \mathfrak{g}^*$ by setting $\chi(\mathfrak{g}_{\bar{1}}) = 0$. Denote the centralizer of χ in $\mathfrak{g}$, which is clearly $\mathbb{Z}_2$-graded, by $\mathfrak{g}_\chi = \mathfrak{g}_{\chi,\bar{0}} + \mathfrak{g}_{\chi,\bar{1}}$. We show that $\dim \mathfrak{g}_{\bar{0}} - \dim \mathfrak{g}_{\chi,\bar{0}} = \dim \mathfrak{g}_{\bar{1}} - \dim \mathfrak{g}_{\chi,\bar{1}}$, and this number is actually an even integer, say $2d$. The following theorem should be regarded as a queer generalization of the celebrated Kac–Weisfeiler conjecture (Premet's theorem [13]) for Lie algebras of reductive algebraic groups.

Theorem C (*Theorem 4.4*)**.** *Let $\chi \in \mathfrak{g}_{\bar{0}}^*$ be a nilpotent p-character. Then the dimension of every simple $U_\chi(\mathfrak{g})$-module is divisible by $\delta = p^d 2^d$.*

The proof of the above theorem is similar to the one in [18] for basic classical Lie superalgebras, which in turn is a generalization of the approach of [13] with some modification using an idea from Skryabin [17]. A $\mathbb{Z}$-grading on $\mathfrak{g}$ associated with χ is first constructed, which leads to the construction of a p-nilpotent Lie subalgebra $\mathfrak{m}$ of $\mathfrak{g}$. Then an elementary argument without using support varieties shows that every simple $U_\chi(\mathfrak{g})$-module is free over $U_\chi(\mathfrak{m})$, and the above theorem follows now on noting that the dimension of $U_\chi(\mathfrak{m})$ is $\delta = p^d 2^d$.

The algebra $U_\chi(\mathfrak{m})$ has a unique simple module K_χ which is one-dimensional. An extra bonus of the above proof is the introduction of a K-superalgebra $W_\chi(\mathfrak{g})$ which will be called the finite W-superalgebra of type Q (see [14,18] for the finite W-(super)algebras associated with the basic classical Lie superalgebras including Lie algebras of reductive algebraic groups). We show further that $U_\chi(\mathfrak{g})$ is isomorphic to the matrix algebra $M_\delta(W_\chi(\mathfrak{g})^{\mathrm{op}})$, and this provides a conceptual explanation of the above δ-divisibility theorem. The complex counterpart of the algebra $W_\chi(\mathfrak{g})$ is expected to have a rich representation theory and will be studied elsewhere [21].

It is worth mentioning that Boe, Kujawa and Nakano have a similar 2-divisibility result for $\mathfrak{g}$-modules in characteristic zero [1].

1.5. For $n = 2$, we are able to analyze in detail the structures of the baby Verma modules and the reduced enveloping algebras for $\mathfrak{q}(2)$. In various cases, we work out the structures of projective covers and the blocks of $U_\chi(\mathfrak{q}(2))$-modules in terms of quivers. Remarkably, the $\mathfrak{q}(2)$ case is far more involved than the classical case of $\mathfrak{sl}(2)$ [5,11] and the $\mathfrak{osp}(1|2)$ case treated in [18].

Let $\chi = \chi_s + \chi_n$ be a Jordan decomposition of a general p-character χ. In contrast to the cases for the simple Lie algebras and basic classical Lie superalgebras, it is not possible to regard the centralizer $\mathfrak{g}_{\chi_s}$ of χ_s in $\mathfrak{g}$ as a Levi subalgebra and to fit it as a middle term of a triangular decomposition of $\mathfrak{g}$. Hence there is no natural generalization to $\mathfrak{g}$ of the functors which give rise to the Morita equivalence [5] which reduces the study of modular representations of a reductive Lie algebra with a general p-character to those of a Levi subalgebra with a nilpotent p-character (also see [18] for a generalization to basic classical Lie superalgebras). Nevertheless, we formulate a conjecture on the existence of a puzzling Morita "super-equivalence" for $\mathfrak{g}$ with its centralizers (without good candidates for adjoint functors), and prove it in the case of $\mathfrak{q}(2)$ by ad hoc direct computations; see Section 7.2 for a precise definition of Morita super-equivalence.

Theorem D (*Theorem 7.5*). *Let $\chi \in \mathfrak{q}(2)^*_{\bar{0}}$ and $\chi = \chi_s + \chi_n$ be its Jordan decomposition. Then the superalgebras $U_\chi(\mathfrak{q}(2))$ and $U_\chi(\mathfrak{q}(2)_{\chi_s})$ are Morita super-equivalent.*

1.6. The paper is organized as follows. In Section 2, we set up some basic notation and constructions on the queer Lie superalgebra $\mathfrak{g}$. In Section 3, we establish our results on the irreducibility of the baby Verma modules and the semisimplicity of the algebra $U_\chi(\mathfrak{g})$. The super-KW property and related construction are presented in Section 4. Sections 5 and 6 are devoted to a detailed study of representations of $\mathfrak{q}(2)$. In Section 7 we formulate a conjecture of Morita super-equivalence for $\mathfrak{q}(n)$ with general p-characters and prove it for $\mathfrak{q}(2)$.

Convention: By subalgebras, ideals, modules, and submodules etc. we mean in the 'super' sense unless otherwise specified. The graded dimension of a superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ will be denoted by $\underline{\dim}\, V = \dim V_{\bar{0}} | \dim V_{\bar{1}}$.

2. The preliminaries

2.1. The Lie superalgebra $\mathfrak{q}(n)$

Let K be an algebraically closed field of characteristic $p > 2$. Let $K^{m|n}$ denote the superspace of dimension $m|n$, and $\mathfrak{gl}(m|n)$ the Lie superalgebra of linear transformations of $K^{m|n}$. Choosing a homogeneous basis of $K^{m|n}$ we may regard $\mathfrak{gl}(m|n)$ as the superalgebra of $(m+n) \times (m+n)$ matrices. In the case where $m = n$ consider an odd automorphism $P : K^{n|n} \to K^{n|n}$ with $P^2 = -1$. The linear transformations of $\mathfrak{gl}(n|n)$ preserving P constitute a subalgebra of $\mathfrak{gl}(n|n)$, denoted by $\mathfrak{q}(n)$. We have $\mathfrak{q}(n) = \mathfrak{q}(n)_{\bar{0}} \oplus \mathfrak{q}(n)_{\bar{1}}$, with $\mathfrak{q}(n)_{\bar{0}}$ isomorphic to the general linear Lie algebra $\mathfrak{gl}(n)$ and $\mathfrak{q}(n)_{\bar{1}}$ isomorphic to the adjoint module of $\mathfrak{gl}(n)$. Choosing P to be the $2n \times 2n$ matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

with I_n denoting the identity $n \times n$ matrix, we may identify $\mathfrak{q}(n)$ as the subalgebra of $\mathfrak{gl}(n|n)$ consisting of $2n \times 2n$ matrices of the form (1.1). The even elements of $\mathfrak{q}(n)$ are those for which $B = 0$, while the odd elements are those for which $A = 0$.

From now on set $\mathfrak{g} = \mathfrak{q}(n)$. The Lie superalgebra $\mathfrak{g}$ admits an odd non-degenerate $\mathfrak{g}$-invariant symmetric bilinear form, which is given by

$$(x, y) := \mathrm{otr}(xy) \quad \text{for } x, y \in \mathfrak{g},$$

where xy denotes the matrix product, and otr denotes the odd trace given by

$$\mathrm{otr}\begin{pmatrix} A & B \\ B & A \end{pmatrix} = \mathrm{trace}(B).$$

It is known that all Cartan subalgebras of $\mathfrak{g}$ are conjugate to the Lie superalgebra $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$ of matrices (1.1) with both A and B diagonal (which will be referred to as the standard Cartan form). All Borel (i.e. maximal solvable) Lie subalgebras of $\mathfrak{g}$ are conjugate to the standard Borel subalgebra consisting of matrices (1.1) with both A and B upper triangular. The roots of $\mathfrak{g}$ (i.e. elements $\alpha \in \mathfrak{h}^*_{\bar{0}}$ for which $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} | \ [h, x] = \alpha(h)x, \forall h \in \mathfrak{h}_{\bar{0}}\} \neq 0$) are the same as $\mathfrak{gl}(n)$: if we let $\{\varepsilon_i\}$ be a basis of $\mathfrak{h}^*_{\bar{0}}$ dual to the standard basis $\{h_i\}$ of $\mathfrak{h}_{\bar{0}}$, where h_i is of the form (1.1) with the ith diagonal entry of A being 1 and 0 elsewhere, then the roots are

$$\Delta = \{\varepsilon_i - \varepsilon_j | \ 1 \le i \neq j \le n\}.$$

The dimension of each root space is equal to $1|1$, in contrast to the $\mathfrak{gl}(n)$ case.

Let us fix some notation. In various places in this paper, we need to work with some fixed Borel subalgebra $\mathfrak{b}$. It determines a system of positive roots which will be denoted by Δ^+; the corresponding simple system is denoted by Π. Also let $\mathfrak{n}^+ = \mathfrak{n}^+_{\bar{0}} + \mathfrak{n}^+_{\bar{1}}$ (respectively $\mathfrak{n}^-$) denote the Lie subalgebra of positive (respectively negative) root vectors.

2.2. The reduced enveloping algebras

Recall that (cf. e.g. [18]) a restricted Lie superalgebra is a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ whose even subalgebra $\mathfrak{g}_{\bar{0}}$ is a restricted Lie algebra with pth power map $^{[p]} : \mathfrak{g}_{\bar{0}} \to \mathfrak{g}_{\bar{0}}$, and the odd part $\mathfrak{g}_{\bar{1}}$ is a restricted $\mathfrak{g}_{\bar{0}}$-module, by the adjoint action. Let $\mathfrak{g}$ be a restricted Lie superalgebra and V be a simple $\mathfrak{g}$-module. The elements $x^p - x^{[p]}$ for $x \in \mathfrak{g}_{\bar{0}}$ in the universal enveloping algebra $U(\mathfrak{g})$ are central. Thus by Schur's lemma, they act as scalars $\zeta(x)$ on V, which can be written as $\chi_V(x)^p$ for some $\chi_V \in \mathfrak{g}^*_{\bar{0}}$. We call χ_V the *p-character* of the module V.

Fix $\chi \in \mathfrak{g}^*_{\bar{0}}$. Let I_χ be the ideal of $U(\mathfrak{g})$ generated by the even central elements $x^p - x^{[p]} - \chi(x)^p$ for all $x \in \mathfrak{g}_{\bar{0}}$. The quotient algebra $U_\chi(\mathfrak{g}) := U(\mathfrak{g})/I_\chi$ is called *the reduced enveloping superalgebra* with p-character χ. A $\mathfrak{g}$-module with p-character χ is the same as a $U_\chi(\mathfrak{g})$-module. We often consider $\chi \in \mathfrak{g}^*$ by letting $\chi(\mathfrak{g}_{\bar{1}}) = 0$.

Recall $\mathfrak{g}_{\bar{0}} = \mathfrak{gl}(n)$. Any p-character $\tilde{\chi}$ is GL(n)-conjugate to a p-character χ with $\chi(\mathfrak{n}_{\bar{0}}^{+}) = 0$, and $U_{\tilde{\chi}}(\mathfrak{g}) \cong U_{\chi}(\mathfrak{g})$. This allows us to restrict ourselves for the rest of the paper to consider only p-characters χ with $\chi(\mathfrak{n}_{\bar{0}}^{+}) = 0$. A p-character $\chi \in \mathfrak{g}_{\bar{0}}^{*}$ is called *semisimple* if it is GL(n)-conjugate to some $\xi \in \mathfrak{g}_{\bar{0}}^{*}$ with $\xi(\mathfrak{n}_{\bar{0}}^{+}) = \xi(\mathfrak{n}_{\bar{0}}^{-}) = 0$, and χ is called *nilpotent* if it is GL(n)-conjugate to some $\eta \in \mathfrak{g}_{\bar{0}}^{*}$ with $\eta(\mathfrak{n}_{\bar{0}}^{+}) = \eta(\mathfrak{h}_{\bar{0}}) = 0$. This could also be viewed alternatively as follows: the odd bilinear form $(,)$ allows one to identify $\mathfrak{g}_{\bar{0}}^{*}$ with $\mathfrak{g}_{\bar{1}}$ which has the same underlying space as $\mathfrak{gl}(n)$. Then the p-character χ is semisimple (respectively nilpotent) if and only if the corresponding element in $\mathfrak{gl}(n)$ is semisimple (respectively nilpotent).

2.3. The baby Verma modules

Fix a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^{-} \oplus \mathfrak{h} \oplus \mathfrak{n}^{+},$$

and let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^{+}$. For $\lambda \in \mathfrak{h}_{\bar{0}}^{*}$ we may consider the symmetric bilinear form on $\mathfrak{h}_{\bar{1}}$ defined by

$$(a|b)_{\lambda} := \lambda([a, b]), \quad a, b \in \mathfrak{h}_{\bar{1}}.$$

Now if $\mathfrak{h}_{\bar{1}}' \subset \mathfrak{h}_{\bar{1}}$ is a maximal isotropic subspace with respect to this bilinear form and we let $\mathfrak{h}_{\bar{1}}''$ be a complement of $\mathfrak{h}_{\bar{1}}'$ in $\mathfrak{h}_{\bar{1}}$ (i.e. $\mathfrak{h}_{\bar{1}} = \mathfrak{h}_{\bar{1}}' \oplus \mathfrak{h}_{\bar{1}}''$), we may extend λ to a one-dimensional representation K_{λ} of $\mathfrak{h}_{\bar{0}} + \mathfrak{h}_{\bar{1}}'$ by letting $\mathfrak{h}_{\bar{1}}'$ act trivially.

Let $\chi \in \mathfrak{g}_{\bar{0}}^{*}$ be such that $\chi(\mathfrak{n}_{\bar{0}}^{+}) = 0$. Set

$$\begin{aligned} \Lambda_{\chi} &= \{\lambda \in \mathfrak{h}_{\bar{0}}^{*} \mid \lambda(h)^{p} - \lambda(h^{[p]}) = \chi(h)^{p} \text{ for all } h \in \mathfrak{h}_{\bar{0}}\} \\ &= \{(\lambda_1, \ldots, \lambda_n) \mid \lambda_i^{p} - \lambda_i = \chi(h_i)^{p},\ 1 \leq i \leq n\}, \end{aligned} \tag{2.1}$$

where $\lambda_i = \lambda(h_i)$. The module K_{λ} is a $U_{\chi}(\mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}')$-module if and only if $\lambda \in \Lambda_{\chi}$. We define an irreducible $U_{\chi}(\mathfrak{h})$-module

$$V_{\chi}(\lambda) = U_{\chi}(\mathfrak{h}) \otimes_{U_{\chi}(\mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}')} K_{\lambda}, \qquad \lambda \in \Lambda_{\chi}.$$

This module has an odd automorphism (or, say, is of type Q) if and only if the dimension of the quotient space $\mathfrak{h}_{\bar{1}}/\ker(\cdot|\cdot)_{\lambda}$ is odd. We extend this irreducible $U_{\chi}(\mathfrak{h})$-module to an irreducible $U_{\chi}(\mathfrak{b})$-module by letting $\mathfrak{n}^{+}$ act trivially. Inducing further we obtain the *baby Verma module* of $U_{\chi}(\mathfrak{g})$ associated with $\lambda \in \mathfrak{h}_{\bar{0}}^{*}$:

$$Z_{\chi}(\lambda) = U_{\chi}(\mathfrak{g}) \otimes_{U_{\chi}(\mathfrak{b})} V_{\chi}(\lambda).$$

We define $v_0 = 1 \otimes 1 \in Z_{\chi}(\lambda)$. We have, as a vector space,

$$Z_{\chi}(\lambda) \cong U_{\chi}(\mathfrak{n}^{-}) \otimes V_{\chi}(\lambda).$$

2.4. $U_{\chi}(\mathfrak{g})$ as symmetric algebra

Recall that the supertrace of an endomorphism X on a vector space $V_{\bar{0}} \oplus V_{\bar{1}}$ is defined to be $\mathrm{str}(X) = \mathrm{tr}(X|_{V_{\bar{0}}}) - \mathrm{tr}(X|_{V_{\bar{1}}})$. An associative superalgebra A with a supersymmetric non-degenerate bilinear form will be called a *symmetric (super)algebra*. One checks that $\mathrm{str}(\mathrm{ad}\, x) = 0$, for all $x \in \mathfrak{g}_{\bar{0}}$. Thus a variant of [18, Prop. 2.7] (also see [5]) applies and it gives the following.

Proposition 2.1. *The superalgebra $U_{\chi}(\mathfrak{g})$ is symmetric for $\chi \in \mathfrak{g}_{\bar{0}}^{*}$.*

3. Modular representations with semisimple *p*-characters

Throughout this section, we assume that $\chi \in \mathfrak{g}_{\bar{0}}^{*}$ is semisimple with $\chi(\mathfrak{n}_{\bar{0}}^{+}) = \chi(\mathfrak{n}_{\bar{0}}^{-}) = 0$. The goal of this section is to establish criteria for the irreducibility of the baby Verma module $Z_{\chi}(\lambda)$ and for the semisimplicity of the algebra $U_{\chi}(\mathfrak{g})$.

3.1. Some $\mathfrak{q}(2)$ calculations

Let us fix some notation for $\mathfrak{q}(2)$ first. We consider the standard generators of $\mathfrak{q}(2)$: $e, f, h_1, h_2, E, F, H_1, H_2$, described symbolically as

$$\left(\begin{array}{cc|cc} h_1 & e & H_1 & E \\ f & h_2 & F & H_2 \\ \hline H_1 & E & h_1 & e \\ F & H_2 & f & h_2 \end{array}\right).$$

This description can be read in the following way: to each symbol there corresponds a matrix of 0's and 1's, in which the 1's are situated precisely at the places occupied by the corresponding symbol. For $n, m \geq 0$, and $k \geq 1$, define

$$T_{n,k} = \binom{n}{k} \frac{(n-1)!}{(k-1)!}, \qquad [x]_m = x(x-1)\cdots(x-m+1).$$

Lemma 3.1. *For $0 \le a \le p-1$, the following identity holds in $U_\chi(\mathfrak{q}(2))$:*

$$e^a f = ae^{a-1}(h_1 - h_2 + a - 1) + fe^a,$$
$$ef^a = af^{a-1}(h_1 - h_2 - (a-1)) + f^a e,$$
$$e^a f^{a-1} = T_{a,1}e[h_1 - h_2 + 1]_{a-1} + T_{a,2}fe^2[h_1 - h_2 + 1]_{a-2} + \cdots + T_{a,a}f^{a-1}e^a, \tag{3.1a}$$
$$e^a F = ae^{a-1}(H_1 - H_2) + (a-1)ae^{a-2}E + Fe^a. \tag{3.1b}$$

Proof. Follows by induction on a. The proof of (3.1a) uses the earlier formulas. □

Lemma 3.2. *The following identity holds in the $U_\chi(\mathfrak{q}(2))$-module $Z_\chi(\lambda)$:*

$$e^{p-1}Ef^{p-1}Fv_0 = (p-1)![h_1 - h_2 - 1]_{p-1}(h_1 + h_2)v_0,$$

where $v_0 = 1 \otimes 1 \in Z_\chi(\lambda)$.

Proof. By a direct computation, we obtain the following identity in $U_\chi(\mathfrak{q}(2))$:

$$Ef^{p-1}F = f^{p-2}F(H_1 - H_2) + f^{p-1}(h_1 + h_2) - f^{p-1}FE.$$

Applying this to the high weight vector v_0 gives us

$$e^{p-1}Ef^{p-1}Fv_0 = e^{p-1}f^{p-2}F(H_1 - H_2)v_0 + e^{p-1}f^{p-1}(h_1 + h_2)v_0.$$

We shall compute the two summands on the right hand side. Using (3.1a) in the second identity below, we have

$$\begin{aligned} e^{p-1}f^{p-1}(h_1 + h_2)v_0 &= (h_1 + h_2)e^{p-1}f^{p-1}v_0 \\ &= \big((h_1 + h_2)(T_{p-1,1}e[h_1 - h_2 + 1]_{p-2}fv_0 + we^2fv_0\big) \\ &= T_{p-1,1}(h_1 + h_2)ef[h_1 - h_2 - 1]_{p-2}v_0 + 0 \\ &= T_{p-1,1}(h_1 + h_2)[(h_1 - h_2) + fe][h_1 - h_2 - 1]_{p-2}v_0 \\ &= T_{p-1,1}(h_1 + h_2)[h_1 - h_2]_{p-1}v_0, \end{aligned}$$

where w is some vector in $U_\chi(\mathfrak{q}(2))$. On the other hand, we have by (3.1a-b) that

$$\begin{aligned} e^{p-1}f^{p-2}F(H_1 - H_2)v_0 &= T_{p-1,1}e[h_1 - h_2 + 1]_{p-2}F(H_1 - H_2)v_0 + ue^2F(H_1 - H_2)v_0 \\ &= T_{p-1,1}eF[h_1 - h_2 - 1]_{p-2}(H_1 - H_2)v_0 + u[2E + 2e(H_1 - H_2) + Fe^2](H_1 - H_2)v_0 \\ &= T_{p-1,1}[(H_1 - H_2) + Fe][h_1 - h_2 - 1]_{p-2}(H_1 - H_2)v_0 \\ &= T_{p-1,1}[h_1 - h_2 - 1]_{p-2}(h_1 + h_2)v_0, \end{aligned}$$

where u is some vector in $U_\chi(\mathfrak{q}(2))$.

It follows by the definition that $T_{p-1,1} = (p-1)!$ and $[x]_{p-1} + [x-1]_{p-2} = [x-1]_{p-1}$. Now the lemma follows from combining the above two computations. □

Define a polynomial ϕ in two variables x, y as follows:

$$\phi(x, y) = (x + y)(x - y - 1)(x - y - 2)\cdots(x - y - (p-1)). \tag{3.1}$$

Proposition 3.3. *Let $\mathfrak{g} = \mathfrak{q}(2)$. Assume that $\chi \in \mathfrak{g}_{\bar{0}}^*$ is semisimple, satisfying $\chi(e) = \chi(f) = 0$, and let $\lambda = (\lambda_1, \lambda_2) \in \Lambda_\chi$. Then the baby Verma module $Z_\chi(\lambda)$ is simple if and only if $\phi(\lambda_1, \lambda_2) \neq 0$.*

Proof. We use the special case of Lemma 3.5 for $n = 2$ (which can also be proved directly) asserting that any non-trivial submodule of $Z_\chi(\lambda)$ contains the vector $f^{p-1}Fv_0$. Now by Lemma 3.2, $Z_\chi(\lambda)$ is simple if and only if $e^{p-1}Ef^{p-1}Fv_0$ is a non-zero multiple of v_0, if and only if $\phi(\lambda_1, \lambda_2) \neq 0$. □

3.2. An irreducibility criterion of $Z_\chi(\lambda)$

We return to the general case for $\mathfrak{g} = \mathfrak{q}(n)$. For the rest of this section, $\mathfrak{h}$ will be the standard Cartan subalgebra with basis $\{h_i, H_i\}_{1\le i\le n}$. Recall that Δ^+ is the set of positive roots associated with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, and that the definition of $Z_\chi(\lambda)$ depends on Δ^+. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda_\chi$ with $\lambda_i = \lambda(h_i)$, put

$$\Phi(\lambda) := \prod_{1\le i<j\le n} \phi(\lambda_i, \lambda_j). \tag{3.2}$$

Theorem 3.4. *Assume that $\chi \in \mathfrak{g}_{\bar{0}}^*$ is semisimple with $\chi(\mathfrak{n}_{\bar{0}}^+) = \chi(\mathfrak{n}_{\bar{0}}^-) = 0$ and let $\lambda \in \Lambda_\chi$. Then the baby Verma module $Z_\chi(\lambda)$ is simple if and only if $\Phi(\lambda) \neq 0$.*

We need some preparations for the proof of the theorem. The height of a root $\alpha \in \Delta^+$ is the sum of the coefficients in the decomposition of α into simple roots. We index the positive roots $\alpha_1, \ldots, \alpha_N$, where $N = n(n-1)/2$, enumerating first the roots of height 1, then the roots of height 2, and so on.

For $\alpha = \varepsilon_k - \varepsilon_l$ $(k < l)$, we use the notation e_α (respectively E_α) for the element of the form (1.1) with the (k, l)-entry of A (respectively B) being 1 and 0 otherwise; also write f_α (respectively F_α) for the element of the form (1.1) with the (l, k)-entry of A (respectively B) being 1 and 0 elsewhere. Further define $e_i = e_{\alpha_i}$ (respectively $f_i = f_{\alpha_i}$) and $E_i = E_{\alpha_i}$ (respectively $F_i = F_{\alpha_i}$). Recall that $N = n(n-1)/2$.

Lemma 3.5. *Any non-zero submodule of a baby Verma module $Z_\chi(\lambda)$ contains the vector $f_1^{p-1}F_1f_2^{p-1}F_2\cdots f_N^{p-1}F_Nv_0$.*

Proof. The proof is similar to that of [15, Proposition 4]. For the sake of the reader, we outline the main steps. We show first that

$$f_j \cdot f_1^{i_1}F_1^{\epsilon_1}\cdots f_{j-1}^{i_{j-1}}F_{j-1}^{\epsilon_{j-1}}f_j^{p-1}F_j^{\epsilon_j}f_{j+1}^{p-1}F_{j+1}\cdots f_N^{p-1}F_Nv_0 = 0,$$
$$F_j \cdot f_1^{i_1}F_1^{\epsilon_1}\cdots f_{j-1}^{i_{j-1}}F_{j-1}^{\epsilon_{j-1}}f_j^{i_j}F_jf_{j+1}^{p-1}F_{j+1}\cdots f_N^{p-1}F_Nv_0 = 0,$$

where $0 \le i_s \le p-1$ and $\epsilon_s = 0, 1$. Then we show that

$$f_j \cdot f_1^{i_1}F_1^{\epsilon_1}\cdots f_j^{i_j-1}F_j^{\epsilon_j}f_{j+1}^{p-1}F_{j+1}\cdots f_N^{p-1}F_Nv_0 = f_1^{i_1}F_1^{\epsilon_1}\cdots f_j^{i_j}F_j^{\epsilon_j}f_{j+1}^{p-1}F_{j+1}\cdots f_N^{p-1}F_Nv_0,$$
$$F_j \cdot f_1^{i_1}F_1^{\epsilon_1}\cdots f_j^{i_j}f_{j+1}^{p-1}F_{j+1}\cdots f_N^{p-1}F_Nv_0 = \pm f_1^{i_1}F_1^{\epsilon_1}\cdots f_j^{i_j}F_jf_{j+1}^{p-1}F_{j+1}\cdots f_N^{p-1}F_Nv_0,$$

where $0 \le i_s \le p-1$ and $\epsilon_s = 0, 1$ for $s \le j$. Now the lemma follows easily from the above claims. $\square$

A PBW basis for a baby Verma module $Z_\chi(\lambda)$ is given by

$$f_1^{a_1}F^{\epsilon_1}\cdots f_N^{a_N}F_N^{\epsilon_N}Y_1^{\tau_1}\cdots Y_r^{\tau_r}v_0 \quad (0 \le a_i \le p-1;\ \epsilon_j, \tau_k = 0, 1)$$

where $\{Y_1, \ldots, Y_r\}$ is a basis for $\mathfrak{h}''_{\bar{1}}$ which we recall is a complement of $\mathfrak{h}'_{\bar{1}}$ in $\mathfrak{h}_{\bar{1}}$. Let $\Lambda(\mathfrak{h}''_{\bar{1}})_+$ be the linear span of $Y_1^{\tau_1}\cdots Y_r^{\tau_r}$, not all $\tau_1, \ldots, \tau_r$ being zero.

Lemma 3.6. *Let $\lambda \in \Lambda_\chi$. The following identity holds in $Z_\chi(\lambda)$:*

$$e_1^{p-1}E_1\cdots e_N^{p-1}E_N \cdot f_1^{p-1}F_1\cdots f_N^{p-1}F_Nv_0 = \tilde{\Phi}(\lambda_1, \ldots, \lambda_n)v_0 + w,$$

for some polynomial $\tilde{\Phi}$ in n variables of degree at most p^N and some $w \in \Lambda(\mathfrak{h}''_{\bar{1}})_+v_0$.

Proof. Follows by a weight consideration and Lemma 3.2. $\square$

Recall that the function ϕ is defined in (3.1).

Lemma 3.7. *Assume that $\alpha = \varepsilon_i - \varepsilon_j$ is a simple root of Δ^+. If $\phi(\lambda_i, \lambda_j) = 0$, then the baby Verma module $Z_\chi(\lambda)$ is reducible.*

Proof. For notational convenience, we assume without loss of generality that $\Delta^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n\}$ and that $\alpha = \varepsilon_1 - \varepsilon_2$. Then we may choose $\mathfrak{h}'_{\bar{1}}, \mathfrak{h}''_{\bar{1}}$ and a basis $\{Y_1, \ldots, Y_r\}$ of $\mathfrak{h}''_{\bar{1}}$ to be compatible with the embedded $\mathfrak{q}(2)$ corresponding to the root $\varepsilon_1 - \varepsilon_2$, so that $\{Y_1\}$ is a basis for $\mathfrak{h}''_{\bar{1}} \cap \mathfrak{q}(2)$ and so that Y_1 is orthogonal to the span $\widetilde{\mathfrak{h}}''_{\bar{1}}$ of $Y_2, \ldots, Y_r$ with respect to $(|)_\lambda$.

Now consider the minimal parabolic subalgebra $\mathfrak{p} = \mathfrak{q}(2) + \mathfrak{b}$, and the induced $\mathfrak{p}$-module $Z_\chi^{\mathfrak{p}}(\lambda) = U_\chi(\mathfrak{p}) \otimes_{U_\chi(\mathfrak{b})} V_\chi(\lambda)$. One can also write $\mathfrak{p} = \mathfrak{q}(2) \oplus \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{n}}^+$, where $\widetilde{\mathfrak{h}}$ is the span of h_i, H_i $(3 \le i \le n)$, and $\widetilde{\mathfrak{n}}^+$ is the span of all positive root vectors except the ones for $\varepsilon_1 - \varepsilon_2$. Note that $[\mathfrak{q}(2), \widetilde{\mathfrak{h}}] = 0$. Since $\phi(\lambda_1, \lambda_2) = 0$, the baby Verma module $Z_\chi^{\mathfrak{q}(2)}(\lambda_1, \lambda_2)$ of $\mathfrak{q}(2)$ is reducible by Proposition 3.3. Then the $\mathfrak{p}$-module $Z_\chi^{\mathfrak{p}}(\lambda)$ is also reducible, thanks to the identification of the $\mathfrak{p}$-modules

$$Z_\chi^{\mathfrak{p}}(\lambda) \cong Z_\chi^{\mathfrak{q}(2)}(\lambda_1, \lambda_2) \otimes \Lambda(\widetilde{\mathfrak{h}}''_{\bar{1}})$$

where the right hand side carries a trivial action of $\widetilde{\mathfrak{n}}^+$. By the transitivity of induced modules we have

$$Z_\chi(\lambda) \cong U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{p})} Z_\chi^{\mathfrak{p}}(\lambda),$$

and then the reducibility of $Z_\chi(\lambda)$ follows from the reducibility of $Z_\chi^{\mathfrak{p}}(\lambda)$. $\square$

Lemma 3.8. *If $\phi(\lambda_i, \lambda_j) = 0$ for some $1 \le i \ne j \le n$, then the baby Verma module $Z_\chi(\lambda)$ is reducible.*

Proof. In this proof, we shall denote Δ^+ and Π by $\Delta^+_{(0)}$ and $\Pi_{(0)}$ respectively, and write $Z^{(0)}_\chi(\lambda) = Z_\chi(\lambda)$. Let $\beta_1 \in \Pi^+_{(0)}$, and $\Delta^+_{(1)} = s_{\beta_1}(\Delta^+_{(0)})$. Let $Z^{(1)}_\chi(\lambda)$ denote the baby Verma module with respect to $\Delta^+_{(1)}$, that is, it is generated by a high weight vector $v^{(1)}_0$ with respect to $\Delta^+_{(1)}$. Then $e^{p-1}_{\beta_1} E_{\beta_1} v^{(1)}_0$ is a weight vector of weight λ, and it is annihilated by e_α, E_α for all $\alpha \in \Pi_{(0)}$. So there is a non-zero $\mathfrak{g}$-homomorphism $\psi_1 : Z^{(0)}_\chi(\lambda) \to Z^{(1)}_\chi(\lambda)$. In general, we can find a sequence of positive roots $\beta_1, \ldots, \beta_t$ such that β_{k+1} $(0 \le k \le t-1)$ is a simple root for the positive system $\Delta^+_{(k)} := s_{\beta_k}(\Delta^+_{(k-1)})$, and such that $\varepsilon_i - \varepsilon_j$ is a simple root for the positive system $\Delta^+_{(t)} = s_{\beta_t}(\Delta^+_{(t-1)})$. By the previous paragraph, there exist non-zero $\mathfrak{g}$-homomorphisms

$$\psi_i : Z^{(i-1)}_\chi(\lambda) \to Z^{(i)}_\chi(\lambda), \quad i = 1, \ldots, t.$$

Since $Z^{(i-1)}_\chi(\lambda)$ and $Z^{(i)}_\chi(\lambda)$, $1 \le i \le t$, have the same dimension, the reducibility of $Z^{(i-1)}_\chi(\lambda)$ follows from the reducibility of $Z^{(i)}_\chi(\lambda)$ via ψ_i. By Lemma 3.7, $Z^{(t)}_\chi$ is reducible, and hence $Z_\chi(\lambda) = Z^{(0)}_\chi(\lambda)$ is also reducible. □

Proof of Theorem 3.4. If $\Phi(\lambda) = 0$, then $\phi(\lambda_i, \lambda_j) = 0$ for some $1 \le i \ne j \le n$. By Lemma 3.8, $Z_\chi(\lambda)$ is reducible. Moreover, by Lemmas 3.5 and 3.6, $\tilde{\Phi}(\lambda) = 0$. Hence, the polynomial $\tilde{\Phi}$ is always divisible by Φ. Conversely, assume that $Z_\chi(\lambda)$ is reducible. By Lemmas 3.5 and 3.6, $\tilde{\Phi}(\lambda) = 0$. Since Φ divides $\tilde{\Phi}$ and $\deg \Phi \ge \deg \tilde{\Phi}$, we conclude that $\Phi(\lambda) = 0$. □

The following corollary is immediate from Theorem 3.4.

Corollary 3.9. *Assume that $\chi \in \mathfrak{g}^*_{\bar 0}$ is semisimple with $\chi(h_1) = \cdots = \chi(h_n)$ (for example, $\chi = 0$). For $\lambda = (a, \ldots, a) \in \Lambda_\chi$ with $a \ne 0$, the baby Verma module $Z_\chi(\lambda)$ is simple.*

3.3. A semisimplicity criterion of $U_\chi(\mathfrak{g})$

Theorem 3.10. *Let χ be semisimple with $\chi(\mathfrak{n}^+_{\bar 0}) = \chi(\mathfrak{n}^-_{\bar 0}) = 0$. The algebra $U_\chi(\mathfrak{g})$ is semisimple if and only if $0 \ne \chi(h_i) \ne \pm\chi(h_j)$ for all $1 \le i \ne j \le n$.*

Proof. Since $\chi(\mathfrak{n}^+_{\bar 0}) = \chi(\mathfrak{n}^-_{\bar 0}) = 0$, each baby Verma module has a unique irreducible quotient, which will be denoted by $L_\chi(\lambda)$. The simple $U_\chi(\mathfrak{g})$-modules $L_\chi(\lambda)$ and $L_\chi(\lambda')$ for $\lambda \ne \lambda'$ are non-isomorphic, and so there are p^n simple $U_\chi(\mathfrak{g})$-modules. By Wedderburn's theorem and a dimension counting argument, $U_\chi(\mathfrak{g})$ is semisimple if and only if all the baby Verma modules $Z_\chi(\lambda)$ for $\lambda \in \Lambda_\chi$ are simple (of type Q for n odd or of type M for n even) and in addition all $\chi(h_i) \ne 0$. Since $\lambda^p_k - \lambda_k = \chi(h_k)^p$ for each k, we have $(\lambda_i \pm \lambda_j)^p - (\lambda_i \pm \lambda_j) = (\chi(h_i) \pm \chi(h_j))^p$.

If $\chi(h_i) \ne \pm\chi(h_j)$ for all $i \ne j$, then every $\lambda \in \Lambda_\chi$ satisfies $\lambda_i \ne -\lambda_j$ and $\lambda_i - \lambda_j \notin \mathbb{F}^*_p$ for all $i \ne j$. So $\Phi(\lambda) \ne 0$, and by Theorem 3.4, $Z_\chi(\lambda)$ is simple for $\lambda \in \Lambda_\chi$.

Conversely, assume $\chi(h_i) = \pm\chi(h_j)$ for some $i \ne j$. If $\chi(h_i) = \chi(h_j)$, then there exists $\lambda \in \Lambda_\chi$ such that $\lambda_i - \lambda_j \in \mathbb{F}^*_p$ (thanks to the flexibility of choosing λ by shifting the value of λ_i by any integer in $\mathbb{F}_p$). If $\chi(h_i) = -\chi(h_j)$, then there exists $\lambda \in \Lambda_\chi$ such that $\lambda_i = -\lambda_j$. In either case, we have $\Phi(\lambda) = 0$. Thus by Theorem 3.4, $Z_\chi(\lambda)$ is reducible. □

4. Modular representations with nilpotent p-characters

4.1. The centralizer of an odd nilpotent element

Let $\chi \in \mathfrak{g}^*_{\bar 0}$ be a p-character, and let $X \in \mathfrak{g}_{\bar 1}$ be such that $\chi = (X, -)$. Then the centralizer $\mathfrak{g}_\chi$ of χ in $\mathfrak{g}$ is identified with the usual centralizer $\mathfrak{g}_X$; that is, $\mathfrak{g}_{X,\bar 0}$ consists of matrices of the form (1.1) with $B = 0$ and A commuting with X (viewed as a matrix), while $\mathfrak{g}_{X,\bar 1}$ consists of matrices of the form (1.1) with $A = 0$ and B anti-commuting with X.

Let $X \in \mathfrak{g}_{\bar 1}$ be nilpotent. Up to a $GL(n)$-conjugation, we can suppose that X has the form (1.1) with $A = 0$ and B equal to the Jordan canonical form

$$B = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_r \end{pmatrix},$$

where J_i is a Jordan block of eigenvalue 0 and size $d_i \times d_i$, and $d_1 \ge d_2 \ge \cdots \ge d_r$.

Proposition 4.1. *Let $X \in \mathfrak{g}_{\bar 1}$ and B be as above, and let $C = (C_{ij})$ be a matrix of the same block type as B. Then:*

(1) *C commutes with B if and only if*

$$C_{ij} = \begin{pmatrix} a & b & \cdots & c \\ & a & \ddots & \vdots \\ & & \ddots & b \\ & & & a \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{for } i \le j, \quad \text{or} \quad C_{ij} = \begin{pmatrix} 0 & a & b & \cdots & c \\ 0 & & a & \ddots & \vdots \\ 0 & & & \ddots & b \\ 0 & & & & a \end{pmatrix} \quad \text{for } i > j.$$

(2) *Also, C anti-commutes with B if and only if*

$$C_{ij} = \begin{pmatrix} a & b & \cdots & c \\ & -a & -b & \vdots \\ & & \ddots & \ddots \\ & & & \pm a \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \textit{for } i \le j, \quad \textit{or} \quad C_{ij} = \begin{pmatrix} 0 & a & b & \cdots & c \\ 0 & & -a & \ddots & \vdots \\ 0 & & & \ddots & \mp b \\ 0 & & & & \pm a \end{pmatrix} \quad \textit{for } i > j.$$

In particular, we have $\dim \mathfrak{g}_{X,\bar{0}} = \dim \mathfrak{g}_{X,\bar{1}} = \sum_{1\le i,j\le r} \min\{d_i, d_j\}$.

Proof. The matrix C commutes with B if and only if

$$J_iC_{ij} = C_{ij}J_j \quad \forall i, j.$$

Also, C anti-commutes with B if and only if

$$J_iC_{ij} = -C_{ij}J_j \quad \forall i, j.$$

Then a direct computation shows that the C_{ij} are of the forms as prescribed in the proposition. The dimension formula for $\dim \mathfrak{g}_{X,i}$ follows. □

4.2. The $\mathbb{Z}$-grading

Let $0 \neq X \in \mathfrak{g}_{\bar{1}}$ be nilpotent. Recall that $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_{\bar{1}} = \mathfrak{gl}(n)$ under the adjoint action of $GL(n)$. Then a standard construction of a $\mathbb{Z}$-grading on $\mathfrak{gl}(n) = \oplus_{k\in\mathbb{Z}}\mathfrak{gl}(n)(k)$ (see [13] and [18, Theorem 3.1]) induces a $\mathbb{Z}$-grading on $\mathfrak{g} = \oplus_{k\in\mathbb{Z}}\mathfrak{g}(k)$ which satisfies $\mathfrak{g}(k) = \mathfrak{g}(k)_{\bar{0}} \oplus \mathfrak{g}(k)_{\bar{1}}$, $\mathfrak{g}(k)_{\bar{0}} = \mathfrak{g}(k)_{\bar{1}} = \mathfrak{gl}(n)(k)$ and the following properties:

$$X \in \mathfrak{g}(2);$$
$$(\mathfrak{g}(k), \mathfrak{g}(l)) = 0, \quad \text{if } k + l \neq 0; \tag{4.1}$$
$$\mathfrak{g}_X = \oplus_{k\in\mathbb{Z}}\mathfrak{g}_X(k) \qquad \text{where } \mathfrak{g}_X(k) = \mathfrak{g}_X \cap \mathfrak{g}(k);$$
$$\mathfrak{g}_X(s) = 0 \qquad \forall s < 0;$$
$$\underline{\dim}\, \mathfrak{g}_X = \underline{\dim}\, \mathfrak{g}(0) + \underline{\dim}\, \mathfrak{g}(1). \tag{4.2}$$

Example 4.2. Let $n = 4$. Let $X \in \mathfrak{g}_{\bar{1}}$ correspond to the Jordan block $J_4 \in \mathfrak{gl}(4)$. The corresponding H is the diagonal matrix diag $(3, 1, -1, -3)$. Then the centralizer $\mathfrak{g}_X$ consists of matrices of the form

$$\left(\begin{array}{cccc|cccc} x_0 & y_2 & z_4 & w_6 & a_0 & b_2 & c_4 & d_6 \\ 0_{-2} & x_0 & y_2 & z_4 & 0_{-2} & -a_0 & -b_2 & -c_4 \\ 0_{-4} & 0_{-2} & x_0 & y_2 & 0_{-4} & 0_{-2} & a_0 & b_2 \\ 0_{-6} & 0_{-4} & 0_{-2} & x_0 & 0_{-6} & 0_{-4} & 0_{-2} & -a_0 \\ \hline a_0 & b_2 & c_4 & d_6 & x_0 & y_2 & z_4 & w_6 \\ 0_{-2} & -a_0 & -b_2 & -c_4 & 0_{-2} & x_0 & y_2 & z_4 \\ 0_{-4} & 0_{-2} & a_0 & b_2 & 0_{-4} & 0_{-2} & x_0 & y_2 \\ 0_{-6} & 0_{-4} & 0_{-2} & -a_0 & 0_{-6} & 0_{-4} & 0_{-2} & x_0 \end{array}\right),$$

where x_i etc. are arbitrary scalars in K, $0_i = 0$, and the index i indicates the $\mathbb{Z}$-gradings of the corresponding matrix entries. Clearly, $\dim \mathfrak{g}_X = 4|4$.

4.3. The super-KW property for nilpotent p-characters

On $\mathfrak{g}(-1)_{\bar{0}}$ (respectively $\mathfrak{g}(-1)_{\bar{1}}$) there is a non-degenerate symplectic (respectively symmetric) bilinear form $\langle\cdot,\cdot\rangle$ given by

$$\langle x, y\rangle := (X, [x, y]) = \chi([x, y]).$$

In other words, the above defines an *even* non-degenerate skew-supersymmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}(-1)$. Indeed, take a non-zero $x \in \mathfrak{g}(-1)_i$ for $i \in \mathbb{Z}_2$. Since $\mathfrak{g}_X(s) = 0$ unless $s \ge 0$, we have that $0 \neq [X, x] \in \mathfrak{g}(1)_{i+\bar{1}}$. By the non-degeneracy of the pairing between $\mathfrak{g}(1)_{i+\bar{1}}$ and $\mathfrak{g}(-1)_i$, there exists $y \in \mathfrak{g}(-1)_i$ with $0 \neq ([X, x], y) = (X, [x, y]) = \langle x, y\rangle$.

Take $\mathfrak{g}(-1)'_i \subset \mathfrak{g}(-1)_i$, where $i \in \mathbb{Z}_2$, to be a maximal isotropic subspace with respect to $\langle\cdot,\cdot\rangle$. Note that $\dim \mathfrak{g}(-1)_i$ is even and $\dim \mathfrak{g}(-1)'_i = \dim \mathfrak{g}(-1)_i/2$. Define a p-nilpotent Lie subalgebra

$$\mathfrak{m} = \bigoplus_{k\ge 2} \mathfrak{g}(-k) \bigoplus \mathfrak{g}(-1)'.$$

Then it follows by (4.1) and (4.2) that

$$\underline{\dim}\, \mathfrak{m} = \frac{1}{2}(\underline{\dim}\, \mathfrak{g} - \underline{\dim}\, \mathfrak{g}_\chi).$$

Proposition 4.3. *Every $U_\chi(\mathfrak{g})$-module is $U_\chi(\mathfrak{m})$-free.*

Proof. The proof is the same as the one for [18, Proposition 4.2], which is in turn a superalgebra generalization of Skryabin [17, Theorem 1.3], and thus is omitted. □

Theorem 4.4 (*Super-KW property with nilpotent characters*)**.** *Let $\chi \in \mathfrak{g}_{\bar{0}}^*$ be nilpotent. Then,* $\dim \mathfrak{g}_{\bar{0}} - \dim \mathfrak{g}_{\chi,\bar{0}} = \dim \mathfrak{g}_{\bar{1}} - \dim \mathfrak{g}_{\chi,\bar{1}}$ *is an even number (denoted by 2d), and the dimension of every simple $U_\chi(\mathfrak{g})$-module is divisible by $\delta = p^d 2^d$.*

Proof. The dimension equality follows from the equality $\dim \mathfrak{g}_{\chi,\bar{0}} = \dim \mathfrak{g}_{\chi,\bar{1}}$ in Proposition 4.1. The divisibility of the dimensions of simple $U_\chi(\mathfrak{g})$-modules is immediate from Proposition 4.3, on noting that $\delta = \dim U_\chi(\mathfrak{m})$. □

Note that $U_\chi(\mathfrak{m})$ has a unique simple module, and this simple module is one-dimensional and will be denoted by K_χ. Denote by $\mathcal{Q}_\mathfrak{m}$ the induced $U_\chi(\mathfrak{g})$-module $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{m})} K_\chi$. We further define the K-superalgebra

$$W_\chi(\mathfrak{g}) = \mathrm{End}_{U_\chi(\mathfrak{g})}(\mathcal{Q}_\mathfrak{m}).$$

Theorem 4.5. (1) *The $U_\chi(\mathfrak{g})$-module $\mathcal{Q}_\mathfrak{m}$ is projective.*
(2) *We have an isomorphism of superalgebras:*

$$U_\chi(\mathfrak{g}) \cong M_\delta(W_\chi(\mathfrak{g})^{op}).$$

Here $M_\delta(W_\chi(\mathfrak{g})^{op})$ denotes the matrix algebra of size δ with entries in $W_\chi(\mathfrak{g})^{op}$.

Proof. The proof is the same as the one for [18, Theorem 4.4], which is a super-generalization (with a mild modification of the proof which bypasses completely the use of support variety) of Premet [14, Theorem 2.3 (i), (ii)], and thus is omitted. □

Remark 4.6. The algebra $W_\chi(\mathfrak{g})$, which is referred to as the finite W-superalgebra of $\mathfrak{q}(n)$, admits a counterpart over the complex field. It will be interesting to develop its structure and representation theory.

5. The representation theory of $\mathfrak{q}(2)$, I

In this and following sections, we study in detail the representation theory of $\mathfrak{g} = \mathfrak{q}(2)$. We still let $\mathfrak{h}$, $\mathfrak{b}$ denote the standard Cartan and Borel subalgebras of $\mathfrak{g}$. Let $\chi \in \mathfrak{q}(2)_{\bar{0}}^*$ be such that $\chi(e) = 0$, but for now we will not impose any condition on $\chi(f)$. In this section, we shall determine the vectors in $Z_\chi(\lambda)$ annihilated by $\mathfrak{n}^+$ for every $\lambda \in \Lambda_\chi$, which is equivalent to describing all possible homomorphisms between baby Verma modules.

5.1. The case where $\lambda = (\lambda_1, \lambda_2) = 0$

In this case we have $\mathfrak{h}'_{\bar{1}} = \mathfrak{h}_{\bar{1}}$. So $Z_\chi(0)$ is induced from the one-dimensional trivial $U_\chi(\mathfrak{b})$-module K_0, and it has a basis $\{f^a F^\epsilon v_0 \mid 0 \le a \le p-1, \epsilon = 0, 1\}$, where we denote by $v_0 = 1 \otimes 1$.

The action of $\mathfrak{g}$ is given by

$$\begin{aligned}
&h.f^aF^\epsilon v_0 = -(a+\epsilon)(\varepsilon_1 - \varepsilon_2)(h) f^a F^\epsilon v_0 \quad \text{for } h \in \mathfrak{h}_{\bar{0}},\\
&H.f^a v_0 = -a(\varepsilon_1 - \varepsilon_2)(H) f^{a-1} F v_0,\\
&H.f^a F v_0 = (\varepsilon_1 + \varepsilon_2)(H) f^{a+1} v_0 \qquad \text{for } H \in \mathfrak{h}_{\bar{1}},\\
&e.f^aF^\epsilon v_0 = -a((a-1) + 2\epsilon) f^{a-1} F^\epsilon v_0,\\
&E.f^aF^\epsilon v_0 = (a-1)a(\epsilon - 1) f^{a-2} F v_0.
\end{aligned}$$

We collect a basis for the vectors annihilated by $\mathfrak{n}^+$ as follows.

<table>
<tr><th>Basis for vectors annihilated by $\mathfrak{n}^+$</th><th>Weights</th></tr>
<tr><td>v_0</td><td rowspan="2">$(0, 0)$</td></tr>
<tr><td>$f^{p-1}F v_0$</td></tr>
<tr><td>$f v_0$</td><td rowspan="2">$(-1, 1)$</td></tr>
<tr><td>$F v_0$</td></tr>
</table>

5.2. The case where $\lambda_1 = \lambda_2 \neq 0$

Take $\mathfrak{h}'_{\bar{1}} = K(H_1+\mu H_2)$, where $\mu \in K$ is such that $\mu^2 = -1$. Then $V_\chi(\lambda)$ is two-dimensional with basis $\{v_0 = 1\otimes 1_\lambda,\ v_1 = H_1 \otimes 1_\lambda\}$, and is of type M. A basis of $Z_\chi(\lambda)$ is given by $\{f^aF^\epsilon \otimes v_i |\ 0 \leq a \leq p-1; \epsilon, i = 0, 1\}$.

We record the action of $\mathfrak{g}$ as follows:

$$
\begin{aligned}
&h \cdot f^aF^\epsilon v_i = (\lambda - (a+\epsilon)\alpha)(h)f^aF^\epsilon v_i\\
&H_1 \cdot f^aFv_0 = -f^aFv_1 + f^{a+1}v_0\\
&H_1 \cdot f^av_0 = f^av_1 - af^{a-1}Fv_0\\
&H_1 \cdot f^aFv_1 = -\lambda_1 f^aFv_0 + f^{a+1}v_1\\
&H_1 \cdot f^av_1 = \lambda_1 f^av_1 - af^{a-1}Fv_1\\
&H_2 \cdot f^aFv_0 = \mu^{-1}f^aFv_1 + f^{a+1}v_0\\
&H_2 \cdot f^av_0 = -\mu^{-1}f^av_1 + af^{a-1}Fv_0\\
&H_2 \cdot f^aFv_1 = \mu\lambda_2 f^aFv_0 + f^{a+1}v_1\\
&H_1 \cdot f^av_1 = -\mu\lambda_2 f^av_1 - af^{a-1}Fv_1\\
&e \cdot f^aFv_0 = [-a(a+1) + a(\lambda_1 - \lambda_2)]f^{a-1}Fv_0 + (1+\mu^{-1})f^av_1\\
&e \cdot f^av_1 = [a(\lambda_1 - \lambda_2) - (a-1)a]f^av_1\\
&e \cdot f^aFv_1 = [-a(a+1) + a(\lambda_1 - \lambda_2)]f^{a-1}Fv_1 + (\lambda_1 + \mu\lambda_2)f^av_0\\
&e \cdot f^av_0 = [a(\lambda_1 - \lambda_2) - (a-1)a]f^{a-1}v_0\\
&E \cdot f^aFv_0 = -a(1+\mu^{-1})f^{a-1}Fv_1 + (\lambda_1 + \lambda_2)f^av_0\\
&E \cdot f^av_1 = -(a-1)af^{a-2}Fv_1 + a(\lambda_1 + \mu\lambda_2)f^{a-1}v_0\\
&E \cdot f^aFv_1 = -a(\lambda_1 + \mu\lambda_2)f^{a-1}Fv_0 + (\lambda_1 + \lambda_2)f^av_1\\
&E \cdot f^av_1 = -(a-1)af^{a-2}Fv_0 + a(1+\mu^{-1})f^{a-1}v_1.
\end{aligned}
$$

A basis for the vectors annihilated by $\mathfrak{n}^+$ is $\{v_0, v_1\}$.

5.3. The case where $\lambda_1 = -\lambda_2 \neq 0$

Take $\mathfrak{h}'_{\bar{1}} = K(H_1 + H_2)$. Then $V_\chi(\lambda)$ is two-dimensional with basis $\{v_0 = 1 \otimes 1_\lambda,\ v_1 = H_1 \otimes 1_\lambda\}$, and is of type M. A basis of $Z_\chi(\lambda)$ is given by $\{f^aF^\epsilon \otimes v_i |\ 0 \leq a \leq p-1; \epsilon, i = 0, 1\}$. The action of $\mathfrak{g}$ is given by the same formula as in Section 5.2, with $\mu = 1$.

A basis for the vectors annihilated by $\mathfrak{n}^+$ is given as follows, where the vectors with weight (λ_2, λ_1) can happen if and only if $\lambda_1 \in \mathbb{F}_p^*$. We use $\star$ in the table here and similar situations below to indicate a conditional existence.

Basis for vectors annihilated by $\mathfrak{n}^+$	Weights
v_0	(λ_1, λ_2)
v_1	
Fv_1	$(\lambda_1 - 1, -\lambda_1 + 1)$
$fv_1 - \lambda_1 Fv_0$	
$f^{2\lambda_1 - 1}Fv_1$	(λ_2, λ_1) $\star$
$4\lambda_1 fv_1 - f^{2\lambda_1}Fv_0$	

5.4. The case where $0 \neq \lambda_1^2 \neq \lambda_2^2 \neq 0$

Let $\mu = -\lambda_1^2/\lambda_2^2$ and $\mathfrak{h}'_{\bar{1}} = K(H_1 + \mu H_2)$. Then $V_\chi(\lambda)$ is two-dimensional with basis $\{v_0 = 1 \otimes 1_\lambda,\ v_1 = H_1 \otimes 1_\lambda\}$, and is of type M. A basis of $Z_\chi(\lambda)$ is given by $\{f^aF^\epsilon \otimes v_i |\ 0 \leq a \leq p-1; \epsilon, i = 0, 1\}$. The action of $\mathfrak{g}$ is given by the same formula as in Section 5.2.

Let b be the integer satisfying $0 \leq b < p$ and $b \equiv \lambda_1 - \lambda_2 - 1 \pmod p$. A basis for the vectors annihilated by $\mathfrak{n}^+$ is given as follows, where the vectors with weight (λ_2, λ_1) can arise if and only if $\lambda_1 - \lambda_2 \in \mathbb{F}_p^*$.

Basis for vectors annihilated by $\mathfrak{n}^+$	Weights
v_0	(λ_1, λ_2)
v_1	
$(b+1)f^bFv_0 - (1+\mu^{-1})f^{b+1}v_1$	(λ_2, λ_1) $\star$
$(b+1)f^bFv_1 - (\lambda_1 + \mu\lambda_2)f^{b+1}v_0$	

5.5. The case where $\lambda_1 = 0, \lambda_2 \neq 0$

Take $\mathfrak{h}'_{\bar{1}} = KH_1$. The irreducible $U_\chi(\mathfrak{b})$-module $V_\chi(\lambda)$ is two-dimensional of type Q with basis $\{v_0 = 1 \otimes 1_\lambda, v_1 = H_2 \otimes 1_\lambda\}$. A basis of $Z_\chi(\lambda)$ is given by $\{f^a F^\epsilon \otimes v_i |\ 0 \leq a \leq p-1; \epsilon, i = 0, 1\}$.

The action of $\mathfrak{g}$ is given by

$$\begin{aligned}
&h \cdot f^a F^\epsilon v_i = (\lambda - (a+\epsilon)\alpha)(h) f^a F^\epsilon v_i \\
&H_1 \cdot f^a F v_0 = f^{a+1} v_0 \\
&H_1 \cdot f^a v_0 = -a f^{a-1} F v_0 \\
&H_1 \cdot f^a F v_1 = f^{a+1} v_1 \\
&H_1 \cdot f^a v_1 = -a f^{a-1} F v_1 \\
&H_2 \cdot f^a F v_0 = -f^a F v_1 + f^{a+1} v_0 \\
&H_2 \cdot f^a v_0 = f^a v_1 + a f^{a-1} F v_0 \\
&H_2 \cdot f^a F v_1 = -\lambda_2 f^a F v_0 + f^{a+1} v_1 \\
&H_2 \cdot f^a v_1 = \lambda_2 f^a v_0 + a f^{a-1} F v_1 \\
&e \cdot f^a F v_0 = [-a(a+1) + -a\lambda_2] f^{a-1} F v_0 - f^a v_1 \\
&e \cdot f^a v_1 = [-a\lambda_2 - (a-1)a] f^a v_1 \\
&e \cdot f^a F v_1 = [-a(a+1) - a\lambda_2] f^{a-1} F v_1 - \lambda_2 f^a v_0 \\
&e \cdot f^a v_0 = [-a\lambda_2 - (a-1)a] f^{a-1} v_0 \\
&E \cdot f^a F v_0 = f^{a-1} F v_1 + \lambda_2 f^a v_0 \\
&E \cdot f^a v_1 = -(a-1)a f^{a-2} F v_1 - a\lambda_2 f^{a-1} v_0 \\
&E \cdot f^a F v_1 = a\lambda_2 f^{a-1} F v_0 + \lambda_2 f^a v_1 \\
&E \cdot f^a v_1 = -(a-1)a f^{a-2} F v_0 - a f^{a-1} v_1.
\end{aligned}$$

Let b be the integer satisfying $0 \leq b < p$ and $b \equiv -\lambda_2 - 1 \pmod p$. A basis for the vectors annihilated by $\mathfrak{n}^+$ is as follows, where the vectors with weight $(\lambda_2, 0)$ can appear in the annihilator if and only if $\lambda_2 \in \mathbb{F}_p^*$.

Basis for vectors annihilated by $\mathfrak{n}^+$	Weights
v_0	$(0, \lambda_2)$
v_1	
$-\lambda_2 f^b F v_0 + f^{b+1} v_1$	$(\lambda_2, 0)$ ⋆
$2(b+1) f^b F v_1 + \lambda_2 f^{b+1} v_0$	

5.6. The case where $\lambda_1 \neq 0, \lambda_2 = 0$

This is similar to 5.5 and is thus omitted.

6. The representation theory of $\mathfrak{q}(2)$, II

In this section, we will study the structures of $U_\chi(\mathfrak{q}(2))$ and its blocks.

Recall that for an associative superalgebra A, a simple A-supermodule N is of type Q (respectively of type M) if $\mathrm{End}_A(N)$ is two-dimensional (respectively one-dimensional), or equivalently if N admits (respectively does not admit) an odd automorphism.

6.1. Structure of $U_\chi(\mathfrak{g})$ for semisimple χ

Assume that χ is semisimple with $\chi(e) = \chi(f) = 0$. We now use the information from Section 5 and in addition that $\chi(f) = 0$ to analyze in detail the structure of $Z_\chi(\lambda)$ and then that of $U_\chi(\mathfrak{g})$.

6.1.1. $0 \neq \chi(h_1)^2 \neq \chi(h_2)^2 \neq 0$

It follows from the results of 5.4 that these baby Verma modules are irreducible of type M, and are pairwise non-isomorphic. By dimension consideration, we conclude that the algebra $U_\chi(\mathfrak{g})$ is semisimple. Of course, this is consistent with Theorem 3.10.

6.1.2. $\chi(h_1) = \chi(h_2) \neq 0$

The high weights $\lambda \in \Lambda_\chi$ are divided into two cases:

(i) $\lambda_1 = \lambda_2$. There are p such weights. The baby Verma modules are as in 5.2, and they are irreducible of type M.

(ii) $\lambda_1 - \lambda_2 \in \mathbb{F}_p^*$. There are $p(p-1)$ such weights. The baby Verma module $Z_\chi(\lambda_1, \lambda_2)$ (see 5.4) is reducible with a unique submodule $L_\chi(\lambda_2, \lambda_1)$ of high weight (λ_2, λ_1) and dimension d, where d is determined by $1 \leq d < 4p$ and $d \equiv 4(\lambda_2 - \lambda_1) \pmod p$. Both the submodule $L_\chi(\lambda_2, \lambda_1)$ and the quotient $L_\chi(\lambda_1, \lambda_2)$ of $Z_\chi(\lambda_1, \lambda_2)$ are irreducible of type M.

The results of Holmes and Nakano [9] apply in our setup, since all the simple modules $L_\chi(\lambda)$ are of type M. In particular, by [9, Thms. 4.5 and 5.1] the projective cover $P_\chi(\lambda)$ of $L_\chi(\lambda)$ has a baby Verma filtration, and for any $\lambda, \mu \in \Lambda_\chi$ one has the Brauer type reciprocity $(P_\chi(\lambda) : Z_\chi(\mu)) = [Z_\chi(\mu) : L_\chi(\lambda)]$, where $(P_\chi(\lambda) : Z_\chi(\mu))$ is the multiplicity of $Z_\chi(\mu)$ appearing in the baby Verma filtration of $P_\chi(\lambda)$, and $[Z_\chi(\mu) : L_\chi(\lambda)]$ is the multiplicity of $L_\chi(\lambda)$ in a composition series of $Z_\chi(\mu)$. Hence:

(i) $P_\chi(\lambda_1, \lambda_1)$ are simple.

(ii) If $\lambda_1 - \lambda_2 \in \mathbb{F}_p^*$, then $(P_\chi(\lambda_1, \lambda_2) : Z_\chi(\mu)) = 1$ for $\mu = (\lambda_1, \lambda_2)$ and (λ_2, λ_1), and 0 otherwise.

Lemma 6.1. *For $\lambda = (\lambda_1, \lambda_2) \in \Lambda_\chi$ with $\lambda_1 - \lambda_2 \in \mathbb{F}_p^*$, the radical series of $P_\chi(\lambda)$ is as follows.*

(1) *head* $P_\chi(\lambda_1, \lambda_2) = \mathit{rad}^2 P_\chi(\lambda_1, \lambda_2) = \mathit{soc}\, P_\chi(\lambda_1, \lambda_2) = L_\chi(\lambda_1, \lambda_2)$.

(2) $\mathit{rad}\, P_\chi(\lambda_1, \lambda_2)/\mathit{rad}^2 P_\chi(\lambda_1, \lambda_2) = L_\chi(\lambda_2, \lambda_1) \oplus L_\chi(\lambda_2, \lambda_1)$.

Proof. Since $U_\chi(\mathfrak{g})$ is a symmetric algebra, we can argue similarly as [7, proof of Proposition 5.1.3]. The details will be omitted here. $\square$

Proposition 6.2. *Let $\mathfrak{g} = \mathfrak{q}(2)$, and let $\chi \in \mathfrak{g}_{\bar{0}}^*$ be semisimple such that $\chi(e) = \chi(f) = 0$ and $\chi(h_1) = \chi(h_2) \neq 0$. Then:*

(i) *For each $(\lambda_1, \lambda_1) \in \Lambda_\chi$, the baby Verma module $Z(\lambda_1, \lambda_1)$ is projective and simple.*

(ii) *For $\lambda = (\lambda_1, \lambda_2) \in \Lambda_\chi$ with $\lambda_1 \neq \lambda_2$, there is a block with exactly two simple modules $L_\chi(\lambda_1, \lambda_2)$ and $L_\chi(\lambda_2, \lambda_1)$, and this block is isomorphic to the algebra given by the quiver*

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, "\alpha"] \arrow[r, bend left=20, "\alpha'"'] & \bullet \arrow[l, bend left=20, "\beta'"'] \arrow[l, bend left=60, "\beta"] \end{tikzcd},$$

with relations $\alpha \circ \beta = \beta \circ \alpha = \alpha' \circ \beta' = \beta' \circ \alpha' = 0$, $\alpha' \circ \beta = \alpha \circ \beta'$, and $\beta' \circ \alpha = \beta \circ \alpha'$.

Proof. Only the last assertion of (ii) on Morita equivalence needs an explanation. The quiver and most of relations can be read off from Lemma 6.1. To get all of the relations, one constructs some projective modules explicitly like in Xiao [20, Section 2.2], then one shows that they are indeed projective covers $P_\chi(\lambda)$. From there, one obtains all relations of the quiver since the homomorphisms between projective covers can be explicitly read off. $\square$

6.1.3. $\chi(h_1) = -\chi(h_2) \neq 0$

The high weights $\lambda \in \Lambda_\chi$ are divided into two cases:

(i) $\lambda_1 = -\lambda_2 \notin \mathbb{F}_p$. There are p such weights. The baby Verma module $Z(\lambda_1, \lambda_2)$ (see 5.3) is reducible with a unique submodule $L_\chi(\lambda_1 - 1, \lambda_2 + 1)$ of dimension $2p$. Both the submodule $L_\chi(\lambda_1 - 1, \lambda_2 + 1)$ and the quotient $L_\chi(\lambda_1, \lambda_2)$ are irreducible of type M.

(ii) $\lambda_1 \neq -\lambda_2$. There are $p(p-1)$ such weights. The baby Verma modules (see 5.4) are irreducible of type M.

Again, the Brauer type reciprocity holds in this case. Hence:

(i) If $\lambda_1 = -\lambda_2 \notin \mathbb{F}_p$, then $(P_\chi(\lambda_1, \lambda_2) : Z_\chi(\mu)) = 1$ for $\mu = (\lambda_1, \lambda_2)$ or $(\lambda_1 + 1, \lambda_2 - 1)$, and is 0 otherwise.

(ii) If $\lambda_1 \neq -\lambda_2$, then $P_\chi(\lambda_1, \lambda_2) = Z_\chi(\lambda_1, \lambda_2) = L_\chi(\lambda_1, \lambda_2)$.

The next lemma follows from this and that $U_\chi(\mathfrak{g})$ is a super-symmetric algebra.

Lemma 6.3. *We have:*

(1) *head* $P_\chi(\lambda_1, -\lambda_1) \cong \mathit{rad}^2 P_\chi(\lambda_1, -\lambda_1) = \mathit{soc}\, P_\chi(\lambda_1, -\lambda_1) = L_\chi(\lambda_1, -\lambda_1)$.

(2) $\mathit{rad}\, P_\chi(\lambda_1, -\lambda_1)/\mathit{rad}^2 P_\chi(\lambda_1, -\lambda_1) \cong L_\chi(\lambda_1 - 1, -\lambda_1 + 1) \oplus L_\chi(\lambda_1 + 1, -\lambda_1 - 1)$.

Proposition 6.4. *Let $\mathfrak{g} = \mathfrak{q}(2)$, and let $\chi \in \mathfrak{g}_{\bar{0}}^*$ be semisimple with $\chi(h_1) = -\chi(h_2) \neq 0$. Then:*

(i) *The p simple modules $L_\chi(\lambda_1, -\lambda_1)$ with $(\lambda_1, -\lambda_1) \in \Lambda_\chi$ are $2p$-dimensional and belong to the same block. This block is isomorphic to the quiver algebra*

$$\begin{tikzcd}
\overset{0}{\bullet} \arrow[r, shift left, "d_0^+"] \arrow[rrrr, bend right=50, shift right, "d_{p-1}^-"'] & \overset{1}{\bullet} \arrow[l, shift left, "d_0^-"] \arrow[r, shift left, "d_1^+"] & \overset{2}{\bullet} \arrow[l, shift left, "d_1^-"] \arrow[r, dotted, no head] & \overset{p-2}{\bullet} \arrow[r, shift left, "d_{p-2}^+"] & \overset{p-1}{\bullet} \arrow[l, shift left, "d_{p-2}^-"] \arrow[llll, bend left=50, "d_{p-1}^+"']
\end{tikzcd},$$

with relations $(d^+)^2 = (d^-)^2 = d^+d^- + d^-d^+ = 0$, where $d^\pm = \sum_{l \in \mathbb{F}_p} d_l^\pm$.

(ii) *Each $Z_\chi(\lambda_1, \lambda_2)$ with $\lambda_1 \neq -\lambda_2$ is projective and simple of type M.*

Proof. The case (ii) is clear by using Brauer reciprocity. Part (i) follows from Lemma 6.3 and an argument similar to that of the proof of [7, Theorem 5.2.1]. $\square$

6.1.4. $\chi(h_1) = 0$, $\chi(h_2) \neq 0$

The high weights $\lambda \in \Lambda_\chi$ are divided into two cases:

(i) $\lambda_1 = 0$. There are p such weights. The baby Verma modules (see 5.5) are irreducible of type Q.
(ii) $\lambda_1 \neq 0$. There are $p(p-1)$ such weights. The baby Verma modules (see 5.4) are irreducible of type M.

Note in both cases we have $Z_\chi(\lambda_1, \lambda_2) = L_\chi(\lambda_1, \lambda_2)$.

The structure theorem of associative superalgebras can be used to estimate the dimensions of projective covers $P_\chi(\lambda_1, \lambda_2)$ of irreducible modules $L_\chi(\lambda_1, \lambda_2)$. To be precise, the dimension of $P_\chi(\lambda_1, \lambda_2)$ equals the number of composition factors of $U_\chi(\mathfrak{g})$ isomorphic to $L_\chi(\lambda_1, \lambda_2)$ if $L_\chi(\lambda_1, \lambda_2)$ is of type M, and equals twice the number if it is type Q. By the exactness of the functor $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{b})} -$, the number of composition factors of $U_\chi(\mathfrak{g})$ isomorphic to $Z_\chi(\lambda_1, \lambda_2)$ equals the number of composition factors of $U_\chi(\mathfrak{b})$ isomorphic to $V_\chi(\lambda_1, \lambda_2)$. This number is $4p$ for all $\lambda \in \Lambda_\chi$.

The dimension of $P_\chi(\lambda_1, \lambda_2)$ is $8p$ in case (i), and is $4p$ in case (ii). In case (i), $P_\chi(\lambda_1, \lambda_2)$ are not simple and they have a simple head $Z_\chi(\lambda_1, \lambda_2) = L_\chi(\lambda_1, \lambda_2)$. On the other hand, $U_\chi(\mathfrak{g})$ is a (super-)symmetric algebra. Thus $P_\chi(\lambda_1, \lambda_2)$ will have $Z_\chi(\lambda_1, \lambda_2) = L_\chi(\lambda_1, \lambda_2)$ as its socle. We conclude that $P_\chi(\lambda_1, \lambda_2)$ is a self-extension of $L_\chi(\lambda_1, \lambda_2)$. As a result, the endomorphism ring $\mathrm{End}_{U_\chi(\mathfrak{g})}(P_\chi(\lambda_1, \lambda_2))$ is isomorphic to the ring $K[x]/\langle x^2 \rangle$, where x corresponds to the projection of $P_\chi(\lambda_1, \lambda_2)$ to its socle. In case (ii), we have $P_\chi(\lambda_1, \lambda_2) = L_\chi(\lambda_1, \lambda_2)$, since they have the same dimension. Put

$$T = \bigoplus_{\lambda_1=0} P_\chi(\lambda_1, \lambda_2) \bigoplus \bigoplus_{\lambda_1 \neq 0} P_\chi(\lambda_1, \lambda_2)^2,$$

where for a module M, M^r denotes the direct sum of r copies of M. The left regular module $U_\chi(\mathfrak{g})$ is isomorphic to T^{2p} and

$$\begin{aligned}
U_\chi(\mathfrak{g}) &\cong \mathrm{End}_{U_\chi(\mathfrak{g})}(U_\chi(\mathfrak{g}))^{op} \cong \mathrm{End}_{U_\chi(\mathfrak{g})}(T^{2p})^{op} \cong (M_{2p}(\mathrm{End}_{U_\chi(\mathfrak{g})}(T)))^{op} \\
&\cong (\oplus_{\lambda_1=0} M_{2p}(\mathfrak{q}_1(K[x]/\langle x^2 \rangle)) \bigoplus \oplus_{\lambda_1 \neq 0} M_{4p}(K))^{op} \\
&\cong (\oplus_{\lambda_1=0} \mathfrak{q}_{2p}(K[x]/\langle x^2 \rangle) \bigoplus \oplus_{\lambda_1 \neq 0} M_{4p}(K))^{op},
\end{aligned}$$

where $\mathfrak{q}_n(K)$ denotes the simple associative superalgebra consisting of all $2n \times 2n$ matrices of the form (1.1). In summary, we have proved the following.

Proposition 6.5. *Let $\mathfrak{g} = \mathfrak{q}(2)$. Let $\chi \in \mathfrak{g}_{\bar{0}}^*$ be semisimple with $\chi(h_1) = 0$ and $\chi(h_2) \neq 0$. Then:*

(i) *Every baby Verma module is irreducible: $Z_\chi(\lambda_1, \lambda_2)$ is of type M for $\lambda_1 \neq 0$, and $Z_\chi(\lambda_1, \lambda_2)$ is of type Q for $\lambda_1 = 0$.*
(ii) *As algebras, $U_\chi(\mathfrak{g}) \cong (\oplus_{\lambda_1=0} \mathfrak{q}_{2p}(K[x]/\langle x^2 \rangle) \bigoplus \oplus_{\lambda_1 \neq 0} M_{4p}(K))^{op}$.*

6.2. Structure of $Z_\chi(\lambda)$ with a mixed p-character

Let $\chi(h_1) = \chi(h_2) \neq 0$, and $\chi(f) = 1$. The high weights $\lambda \in \Lambda_\chi$ are divided into two cases:

(i) $\lambda_1 = \lambda_2$. There are p such weights. The baby Verma modules (see 5.2) are irreducible of type M and pairwise non-isomorphic.
(ii) $\lambda_1 \neq \lambda_2$. There are $p(p-1)$ such weights. The baby Verma modules (see 5.4) are irreducible of type M. We have $Z_\chi(\lambda_1, \lambda_2) \cong Z_\chi(\lambda_2, \lambda_1)$ and there is no other isomorphism among these baby Verma modules.

Arguing similarly to in 6.1.4, we prove the following.

Proposition 6.6. *Let $\mathfrak{g} = \mathfrak{q}(2)$. Let $\chi \in \mathfrak{g}_{\bar{0}}^*$ be such that $\chi(h_1) = \chi(h_2) \neq 0$ and $\chi(f) = 1$. Then:*

(i) *Every baby Verma module is simple, $4p$-dimensional and of type M.*
(ii) *For $(\lambda_1, \lambda_1) \in \Lambda_\chi$, the baby Verma module $Z_\chi(\lambda_1, \lambda_1)$ is projective.*
(iii) *For $(\lambda_1, \lambda_2) \in \Lambda_\chi$ with $\lambda_1 \neq \lambda_2$, the projective cover is a self-extension of $Z_\chi(\lambda_1, \lambda_2)$.*
(iv) *As algebras, $U_\chi(\mathfrak{g}) \cong M_{4p}(K)^{\oplus p} \oplus M_{4p}(K[x]/\langle x^2\rangle)^{\oplus \frac{p(p-1)}{2}}$.*

6.3. Structures of $U_0(\mathfrak{g})$-modules

Let $\chi = 0$. We shall drop the index χ or 0 for the baby Verma, projective and simple modules of $U_0(\mathfrak{g})$.
We artificially divide the baby Verma modules into the following.

(i) $(\lambda_1, \lambda_2) = (0, 0)$. The baby Verma module (see 5.1) has a unique submodule, $L(p-1, 1-p)$, of dimension $(2p-2)$, while the irreducible quotient $L(0, 0)$ is two-dimensional.
(ii) $(\lambda_1, -\lambda_1)$, $\lambda_1 \neq 0$. There are $(p-1)$ such weights. By analyzing the vectors annihilated by $\mathfrak{n}_{\bar{0}}^+$, we see that each baby Verma module $Z(\lambda_1, -\lambda_1)$ (see 5.3) has a composition series of four simple modules:

$$L(\lambda_1, -\lambda_1),\ L(p-1-\lambda_1, 1-p+\lambda_1),\ L(\lambda_1 - 1, -\lambda_1 + 1),\ L(-\lambda_1, \lambda_1).$$

The dimension of $L(\lambda_1, -\lambda_1)$ is the number b determined by the conditions $0 \leq b < 2p$ and $b \equiv (4\lambda_1 - 2) \pmod{2p}$.
(iii) (λ_1, λ_1), $\lambda_1 \neq 0$. There are $(p-1)$ such weights. The baby Verma modules (see 5.2) are simple of type M.
(iv) $(0, \lambda_2)$, $\lambda_2 \neq 0$. There are $(p-1)$ such weights. By examining the vectors annihilated by $\mathfrak{n}_{\bar{0}}^+$, we see that the baby Verma module $Z(0, \lambda_2)$ (see 5.5) has a simple head $L(0, \lambda_2)$ and a simple socle $L(\lambda_2, 0)$, both $2p$-dimensional and of type Q.
(v) $(\lambda_1, 0)$, $\lambda_1 \neq 0$. This case is similar to case (iv), and thus omitted.
(vi) (λ_1, λ_2) with $0 \neq \lambda_1^2 \neq \lambda_2^2 \neq 0$. There are $(p-1)(p-3)$ such weights. Each baby Verma module $Z(\lambda_1, \lambda_2)$ (see 5.4) has a unique submodule $L(\lambda_2, \lambda_1)$, which is simple of dimension d, where d is determined by $0 \leq d < 4p$ and $d \equiv 4(\lambda_2 - \lambda_1) \pmod{p}$. The head $L(\lambda_1, \lambda_2)$ is simple of dimension $4p - d$.

By the exactness of the functor $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{b})} -$, the number of composition factors of $U_\chi(\mathfrak{g})$ isomorphic to $Z(\lambda_1, \lambda_2)$ equals the number of composition factors of $U_\chi(\mathfrak{b})$ isomorphic to $V(\lambda_1, \lambda_2)$. This number is $8p$ for weight $(0, 0)$, and $4p$ otherwise. The structures of baby Verma modules have been given explicitly above. From this we conclude that

$$\dim P(\lambda_1, \lambda_2) = \begin{cases} 16p, & \text{if } \lambda_1 = \lambda_2 = 0; \\ 16p, & \text{if } \lambda_1 = -\lambda_2 \neq 0; \\ 4p, & \text{if } \lambda_1 = \lambda_2 \neq 0; \\ 16p, & \text{if } \lambda_1 = 0, \lambda_2 \neq 0; \\ 16p, & \text{if } \lambda_1 \neq 0, \lambda_2 = 0; \\ 8p, & \text{if } 0 \neq \lambda_1^2 \neq \lambda_2^2 \neq 0. \end{cases}$$

From this we further conclude that $Z(\lambda_1, \lambda_2)$ is projective if $\phi(\lambda_1, \lambda_2) \neq 0$ and $\lambda_i \neq 0$, $i = 1, 2$. In particular, $Z(a, a)$ is projective and simple for $a \in \mathbb{F}_p^*$ as claimed in Theorem 3.4.

6.4. Structure of $U_\chi(\mathfrak{g})$-modules with χ nilpotent

Assume $\chi(e) = \chi(h_1) = \chi(h_2) = 0$, and $\chi(f) = 1$. Since χ is of standard Levi form (cf. [11, Definition 10.1]), each baby Verma module $Z_\chi(\lambda)$ has a unique irreducible quotient $L_\chi(\lambda)$ (cf. [11, Proposition 10.2]; the same argument applies here). As in the (restricted) case where $\chi = 0$, we divide the baby Verma modules according to their high weights as follows:

(i) $(\lambda_1, \lambda_2) = (0, 0)$. The baby Verma module (see 5.1) is simple of type M. We have an isomorphism $L_\chi(0, 0) \cong L_\chi(p-1, 1-p)$.
(ii) $(\lambda_1, -\lambda_1)$, $\lambda_1 \neq 0$. There are $(p-1)$ such weights. By analyzing the vectors annihilated by $\mathfrak{n}_{\bar{0}}^+$ (see 5.3), we see that the baby Verma modules $Z_\chi(\lambda_1, -\lambda_1)$ has a simple socle $L_\chi(-\lambda_1, \lambda_1)$ and a simple head $L_\chi(\lambda_1, -\lambda_1)$, each of dimension $2p$. We have an isomorphism $L_\chi(\lambda_1, -\lambda_1) \cong L_\chi(p-1-\lambda_1, 1-p+\lambda_1)$.
(iii) (λ_1, λ_1), $\lambda_1 \neq 0$. There are $(p-1)$ such weights. The baby Verma modules (see 5.2) are simple of type M.
(iv) $(0, \lambda_2)$, $\lambda_2 \neq 0$. There are $(p-1)$ such weights. The baby Verma modules (see 5.5) are simple of type Q. The vectors annihilated by $\mathfrak{n}_{\bar{0}}^+$ given in 5.5 provide us with an isomorphism $Z_\chi(0, \lambda_2) \cong Z_\chi(\lambda_2, 0)$.
(v) $(\lambda_1, 0)$, $\lambda_1 \neq 0$. This case is similar to case (iv), and the baby Verma modules $Z_\chi(\lambda_1, 0)$ are simple of type Q.
(vi) (λ_1, λ_2) with $0 \neq \lambda_1^2 \neq \lambda_2^2 \neq 0$. There are $(p-1)(p-3)$ such weights. The baby Verma modules (see 5.4) are simple of type M. We have an isomorphism $Z_\chi(\lambda_1, \lambda_2) \cong Z_\chi(\lambda_2, \lambda_1)$.

By the same argument as in the previous subsection, we estimate the dimensions of projective covers $P_\chi(\lambda_1, \lambda_2)$ of $L_\chi(\lambda_1, \lambda_2)$ as follows:

$$\dim P_\chi(\lambda_1,\lambda_2) = \begin{cases} 16p & \text{for } L_\chi(\lambda_1, -\lambda_1) \cong L_\chi(p-1-\lambda_1, 1-p+\lambda_1), \lambda_1 \in \mathbb{F}_p; \\ 8p & \text{for } L_\chi(\frac{p-1}{2}, \frac{p+1}{2}) \cong L_\chi(\frac{p+1}{2}, \frac{p-1}{2}); \\ 4p & \text{for } L_\chi(\lambda_1, \lambda_1), \lambda_1 \neq 0; \\ 16p & \text{for } L_\chi(0, \lambda_2) \cong L_\chi(\lambda_2, 0); \\ 8p & \text{for } L_\chi(\lambda_1, \lambda_2) \cong L_\chi(\lambda_2, \lambda_1), 0 \neq \lambda_1^2 \neq \lambda_2^2 \neq 0. \end{cases}$$

For this, we conclude that $Z_\chi(\lambda_1, \lambda_1)$ are projective simple for $\lambda_1 \neq 0$. Since $U_\chi(\mathfrak{g})$ is a symmetric algebra by Proposition 2.1, $P_\chi(\lambda_1, \lambda_2)$ is a self-extension of $L_\chi(\lambda_1, \lambda_2) = Z_\chi(\lambda_1, \lambda_2)$ for $0 \neq \lambda_1^2 \neq \lambda_2^2 \neq 0$.

7. Modular representations with general *p*-characters

7.1. The centralizer of an odd element

For a general odd element $X \in \mathfrak{g}_{\bar{1}}$, let $X = X_s + X_n$ be its Jordan decomposition (which is understood via the identification $\mathfrak{g}_{\bar{1}} = \mathfrak{gl}(n)$). As in the Lie algebra setup, we clearly have

$$\mathfrak{g}_{X,\bar{0}} = \mathfrak{g}_{X_s,\bar{0}} \cap \mathfrak{g}_{X_n,\bar{0}}.$$

A much less trivial relation holds for the odd parts of the corresponding centralizers.

Lemma 7.1. *Let $X = X_s + X_n$ be the Jordan decomposition of an odd element $X \in \mathfrak{g}_{\bar{1}}$. Then we have*

$$\mathfrak{g}_{X,\bar{1}} = \mathfrak{g}_{X_s,\bar{1}} \cap \mathfrak{g}_{X_n,\bar{1}} \tag{7.1}$$

and thus $\mathfrak{g}_X = \mathfrak{g}_{X_s} \cap \mathfrak{g}_{X_n}$.

Proof. Without lost of generality, we assume that X is of Jordan canonical form

$$X = \begin{pmatrix} J_{d_1}(\lambda_1) & & \\ & \ddots & \\ & & J_{d_r}(\lambda_r) \end{pmatrix},$$

where $J_{d_i}(\lambda_i)$ denotes the $d_i \times d_i$ Jordan block with λ_i on the diagonal. Then by Horn–Johnson [10, Theorem 4.4.11], the dimension of $\mathfrak{g}_{X,\bar{1}}$ is given by

$$\dim \mathfrak{g}_{X,\bar{1}} = \sum_{\lambda_i = -\lambda_j} \min\{d_i, d_j\}.$$

Recall the description of $\mathfrak{g}_{X_n,\bar{1}}$ in Section 4.1. On the other hand, an element in $\mathfrak{g}_{X_s,\bar{1}}$ has the form (1.1) with $A = 0$ and

$$B = \begin{pmatrix} B_{11} & \cdots & B_{1r} \\ \vdots & \ddots & \vdots \\ B_{r1} & \cdots & B_{rr} \end{pmatrix},$$

where the $d_i \times d_j$ matrix $B_{ij} = 0$ if $\lambda_i \neq -\lambda_j$ and B_{ij} is arbitrary if $\lambda_i = -\lambda_j$. It follows that the dimension of $\mathfrak{g}_{X_s,\bar{1}} \cap \mathfrak{g}_{X_n,\bar{1}}$ is $\sum_{\lambda_i=-\lambda_j} \min\{d_i, d_j\}$, which is same as $\dim \mathfrak{g}_{X,\bar{1}}$ given above. Obviously $\mathfrak{g}_{X_s,\bar{1}} \cap \mathfrak{g}_{X_n,\bar{1}} \subseteq \mathfrak{g}_{X,\bar{1}}$, so $\mathfrak{g}_{X,\bar{1}} = \mathfrak{g}_{X_s,\bar{1}} \cap \mathfrak{g}_{X_n,\bar{1}}$. $\square$

Assume now that the odd element $X \in \mathfrak{g}_{\bar{1}}$ is semisimple and hence is $\mathrm{GL}(n)$-conjugate to some element $Y \in \mathfrak{g}_{\bar{1}}$ of the form (1.1) with $A = 0$ and

$$B = \mathrm{diag}(\underbrace{0,\ldots,0}_{m}, \underbrace{\mu_1,\ldots,\mu_1}_{r_1}, \underbrace{-\mu_1,\ldots,-\mu_1}_{s_1}, \ldots, \underbrace{\mu_t,\ldots,\mu_t}_{r_t}, \underbrace{-\mu_t,\ldots,-\mu_t}_{s_t}),$$

where $\mu_1, \ldots, \mu_t$ are squarely distinct non-zero scalars, and $m, r_i, s_i \geq 0$. The next lemma follows by a direct computation.

Lemma 7.2. *For a semisimple odd element $X \in \mathfrak{g}_{\bar{1}}$ as above, the centralizer $\mathfrak{g}_X$ is isomorphic to a direct sum $\mathfrak{q}(m) \oplus \mathfrak{gl}(r_1|s_1) \oplus \cdots \oplus \mathfrak{gl}(r_t|s_t)$.*

7.2. A conjecture of Morita super-equivalence

Given a finite dimensional superalgebra A, we denote by A-$\mathfrak{mod}$ the category of finite dimensional A-modules and $\mathrm{Irr}(A)$ the set of isoclasses of simple A-supermodules.

Conjecture 7.3. *Let $\chi \in \mathfrak{g}_{\bar{0}}^*$ be a p-character with Jordan decomposition $\chi = \chi_s + \chi_n$. Let $b_i = \dim \mathfrak{g}_i - \dim \mathfrak{g}_{\chi_s,i}$ for $i \in \mathbb{Z}_2$. Then there are adjoint exact functors F and G:*

$$U_\chi(\mathfrak{g})\text{-}\mathfrak{mod} \underset{G}{\overset{F}{\rightleftarrows}} U_\chi(\mathfrak{g}_{\chi_s})\text{-}\mathfrak{mod}$$

satisfying the following:

(i) *Suppose b_1 is even. Then F and G are inverse equivalences of categories, inducing a type-preserving bijection between $Irr(U_\chi(\mathfrak{g}))$ and $Irr(U_\chi(\mathfrak{g}_{\chi_s}))$. Moreover, for a $U_\chi(\mathfrak{g}_{\chi_s})$-module V, $\dim G(V) = p^{\frac{b_0}{2}} 2^{\frac{b_1}{2}} \dim V$.*

(ii) *Suppose b_1 is odd. Then*

$$F \circ G \cong Id \oplus \Pi \quad G \circ F \cong Id \oplus \Pi,$$

where Π is the parity change functor in a module category of a superalgebra. The functor F induces a bijection of $Irr(U_\chi(\mathfrak{g}))$ of type M (respectively, of type Q) and $Irr(U_\chi(\mathfrak{g}_{\chi_s}))$ of type Q (respectively, of type M). Moreover, for $V \in Irr(U_\chi(\mathfrak{g}_{\chi_s}))$ of type M, the dimension of the corresponding $U_\chi(\mathfrak{g})$-module $G(V)$ is $p^{\frac{b_0}{2}} 2^{\frac{b_1+1}{2}} \dim V$; while for $V \in Irr(U_\chi(\mathfrak{g}_{\chi_s}))$ of type Q, the dimension of $G(V)$ is $p^{\frac{b_0}{2}} 2^{\frac{b_1-1}{2}} \dim V$.

In the above and later on, χ in $U_\chi(\mathfrak{g}_{\chi_s})$ is understood as the restriction of χ to $\mathfrak{g}_{\chi_s}$. One can show by Lemma 7.2 that

$$b_1 \equiv \#\{1 \le i \le n \mid \chi(h_i) \ne 0\} \mod 2.$$

We will say that the superalgebras $U_\chi(\mathfrak{g})$ and $U_\chi(\mathfrak{g}_{\chi_s})$ are *Morita super-equivalent* if they satisfy the properties prescribed in the above conjecture. In case (i) above, the superalgebras are indeed Morita equivalent in the usual sense.

Remark 7.4. When $\mathfrak{g}$ is one of the basic classical Lie superalgebras, the above Morita super-equivalence is indeed the usual Morita equivalence with explicitly given functors ([18, Theorem 5.2]). This in turn was a generalization of a theorem of Friedlander and Parshall [5] (also cf. [19]) for Lie algebras of reductive algebraic groups. However, $\mathfrak{g} = \mathfrak{q}(n)$ does not admit a natural triangular decomposition with $\mathfrak{g}_{\chi_s}$ as the middle term. This is already evident from the calculation of $\mathfrak{q}(2)$ below. Hence, the natural adjoint functors in [5] (also [18]) have no counterpart in the current setup.

Note that there is a similar result of Frisk and Mazorchuk [6] in characteristic zero which establishes a super-equivalence between the strongly typical blocks of the category $\mathcal{O}$ of $\mathfrak{q}(n)$ and those of its even subalgebra $\mathfrak{gl}(n)$. It is interesting to see whether it is possible to adapt their method to the modular setting.

7.3. The Morita super-equivalence for $\mathfrak{q}(2)$

Theorem 7.5. *Conjecture 7.3 holds for $\mathfrak{q}(2)$. That is, the algebras $U_\chi(\mathfrak{q}(2))$ and $U_\chi(\mathfrak{q}(2)_{\chi_s})$ are Morita super-equivalent.*

This subsection is devoted to the proof of the above theorem by a detailed analysis of the representation theory of the centralizers of the semisimple part χ_s of p-characters χ in the case of $\mathfrak{q}(2)$ and then a comparison with the results in Section 6.

7.3.1. Semisimple χ with $0 \ne \chi(h_1)^2 \ne \chi(h_2)^2 \ne 0$

The centralizer $\mathfrak{g}_{\chi_s}$ is the even Cartan subalgebra $\mathfrak{h}_{\bar{0}}$. The algebra $U_\chi(\mathfrak{h}_{\bar{0}})$ is semisimple and commutative. Thus $U_\chi(\mathfrak{h}_{\bar{0}})$ and $U_\chi(\mathfrak{g})$ are Morita equivalent, by a comparison with 6.1.1.

7.3.2. Semisimple χ with $\chi(h_1) = \chi(h_2) \ne 0$

The centralizer $\mathfrak{g}_{\chi_s}$ is the even subalgebra $\mathfrak{g}_{\bar{0}} = \mathfrak{gl}(2)$. Its reduced enveloping algebra $U_\chi(\mathfrak{g}_{\bar{0}}) = U_\chi(\mathfrak{gl}(2))$ is isomorphic to $U_0(\mathfrak{sl}(2)) \otimes K[x]/(x^p - x - \chi(h_1)^p)$. By combining this with the well-known structure of the algebra $U_0(\mathfrak{sl}(2))$ (see for example [5, Proposition 2.4] and [8, Example 3.10]), we deduce the following.

Proposition 7.6. *Let $\chi \in \mathfrak{gl}(2)_{\bar{0}}^*$ be semisimple with $\chi(h_1) = \chi(h_2) \ne 0$. Then:*

(i) *The baby Verma $U_\chi(\mathfrak{gl}(2))$-module with $\lambda_1 - \lambda_2 = p - 1$ is projective and simple.*

(ii) *For $\lambda_1 - \lambda_2 \in \mathbb{F}_p \backslash \{p-1\}$, there is one block with exactly two simple $U_\chi(\mathfrak{gl}(2))$-modules of high weights (λ_1, λ_2) and $(\lambda_2 - 1, \lambda_1 + 1)$. This block is isomorphic to the quiver algebra given in Proposition 6.2(ii).*

By a comparison with Proposition 6.2, we see that $U_\chi(\mathfrak{g})$ and $U_\chi(\mathfrak{g}_{\chi_s})$ are Morita equivalent.

7.3.3. Semisimple χ with $\chi(h_1) = -\chi(h_2) \neq 0$

The centralizer $\mathfrak{g}_{\chi_s}$ is spanned by h_1, h_2, E, and F and it is isomorphic to the Lie superalgebra $\mathfrak{gl}(1|1)$. Denote by $\tilde{\mathfrak{b}}$ the subalgebra $Kh_1 \oplus Kh_2 \oplus KE$. We have all irreducible $U_\chi(\tilde{\mathfrak{b}})$-modules given by $K_\lambda = K$ with $\lambda(h_i)^p - \lambda(h_i) = \chi(h_i)^p$, upon which h_i acts as a scalar λ_i, and E acts as zero. Inducing from K_λ, we get the baby Verma modules for $U_\chi(\mathfrak{gl}(1|1))$:

$$\tilde{Z}_\chi(\lambda) = U_\chi(\mathfrak{gl}(1|1)) \otimes_{U_\chi(\tilde{\mathfrak{b}})} K_\lambda$$

which is two-dimensional and has a unique simple quotient $\tilde{L}_\chi(\lambda)$.

Proposition 7.7. *Let $\chi \in \mathfrak{gl}(1|1)^*_{\bar{0}}$ be such that $\chi(h_1) = -\chi(h_2) \neq 0$. Then:*

(i) *The p simple modules $\tilde{L}(\lambda_1, -\lambda_1)$ belong to a single block, and this block is isomorphic to the quiver algebra with relations given in Proposition 6.4(ii).*
(ii) *Each baby Verma module $\tilde{Z}_\chi(\lambda_1, \lambda_2)$ with $\lambda_1 + \lambda_2 \neq 0$ is projective and simple.*

Proof. When $\lambda_2 = -\lambda_1$, the baby Verma module $\tilde{Z}_\chi(\lambda_1, -\lambda_1)$ is reducible and has a unique (one-dimensional) submodule of weight $(\lambda_1 - 1, -\lambda_1 + 1)$. A projective cover of $\tilde{L}(\lambda_1, -\lambda_1)$ can be constructed explicitly (this is similar to and simpler than the $\mathfrak{sl}(2)$ case [20]). This leads to the calculation of the underlying block in terms of quivers.

The remaining case with $\lambda_1 + \lambda_2 \neq 0$ is easy. □

By a comparison with Proposition 6.4, we see that $U_\chi(\mathfrak{g})$ and $U_\chi(\mathfrak{g}_{\chi_s})$ are Morita equivalent.

7.3.4. Semisimple χ with $\chi(h_1) = 0$ and $\chi(h_2) \neq 0$

The centralizer $\mathfrak{g}_{\chi_s}$ is $\mathfrak{h}_{\bar{0}} \oplus KH_1 \cong \mathfrak{q}(1) \oplus Kh_2$. The weights $\lambda \in \mathfrak{h}^*_{\bar{0}}$ such that $\lambda(h_i)^p - \lambda(h_i) = \chi(h_i)^p$ can be divided into two cases:

(i) $\lambda_1 = 0$. Then the relations $h_1 v = 0$, $h_2 v = \lambda_2 v$ and $H_1 v = 0$ define a one-dimensional $U_\chi(\mathfrak{g}_{\chi_s})$-module.
(ii) $\lambda_1 \neq 0$. The irreducible $U_\chi(\mathfrak{h}_{\bar{0}})$-modules are one-dimensional of the form Kv upon which h_i act as scalars λ_i. We have the induced $U_\chi(\mathfrak{g}_{\chi_s})$-modules:

$$\tilde{Z}_\chi(\lambda_1, \lambda_2) = U_\chi(\mathfrak{g}_{\chi_s}) \otimes_{U_\chi(\mathfrak{h}_{\bar{0}})} Kv$$

which are irreducible of type Q.

By an analysis parallel to that in 6.1.4, we have an algebra isomorphism

$$U_\chi(\mathfrak{g}_{\chi_s}) \cong \left(\oplus_{\lambda_1=0} M_1(K[x]/\langle x^2 \rangle) \bigoplus \oplus_{\lambda_1 \neq 0} \mathfrak{q}_1(K) \right)^{op}.$$

Recall that the algebra $U_\chi(\mathfrak{g})$ was computed in Proposition 6.5 and it is indeed Morita equivalent to $U_\chi(\mathfrak{g}_{\chi_s})$.

7.3.5. A mixed case: $\chi(h_1) = \chi(h_2) \neq 0$ and $\chi(f) = 1$

The centralizer $\mathfrak{g}_{\chi_s}$ is the even subalgebra $\mathfrak{g}_{\bar{0}} \cong \mathfrak{gl}(2)$. By the identification via $U_\chi(\mathfrak{sl}(2))$ as in 7.3.2, we show that

$$U_\chi(\mathfrak{g}_{\chi_s}) \cong M_p(K)^{\oplus p} \oplus M_p(K[x]/\langle x^2 \rangle)^{\oplus \frac{p(p-1)}{2}}.$$

It follows by comparing with the algebra $U_\chi(\mathfrak{g})$ computed in Proposition 6.6 that $U_\chi(\mathfrak{g})$ and $U_\chi(\mathfrak{g}_{\chi_s})$ are indeed Morita equivalent.

Remark 7.8. Recall that [18] the super-KW conjecture for a restricted Lie superalgebra $\mathfrak{g}$ states that the dimension of every $U_\chi(\mathfrak{g})$-module is divisible by $p^{\frac{d_0}{2}} 2^{\lfloor \frac{d_1}{2} \rfloor}$, where $\lfloor a \rfloor$ denotes the least integer upper bound of a.

Let $\mathfrak{g} = \mathfrak{q}(n)$ and let $\chi = \chi_s + \chi_n$ be the Jordan decomposition of a general p-character χ for the Lie superalgebra $\mathfrak{g}$. Let $d_i = \dim \mathfrak{g}_i - \dim \mathfrak{g}_{\chi,i}$, $i \in \mathbb{Z}_2$. By a comparison of dimensions using (7.1) and Lemma 7.1,

$$\underline{\dim}\,\mathfrak{g} - \underline{\dim}\,\mathfrak{g}_\chi = \underline{\dim}\,\mathfrak{g} - \underline{\dim}\,(\mathfrak{g}_{\chi_s})_{\chi_n} = b_0|b_1 + (\underline{\dim}\,\mathfrak{g}_{\chi_s} - \underline{\dim}\,(\mathfrak{g}_{\chi_s})_{\chi_n}),$$

we see that the super-KW conjecture for $\mathfrak{g}$ would follow from the validity of Conjecture 7.3 when combined with the super-KW property for nilpotent p-character established in Theorem 4.4.

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References

[1] B. Boe, J. Kujawa, D. Nakano, Cohomology and support varieties for Lie superalgebras II, Proc. Lond. Math. Soc. 98 (2009) 19–44.
[2] J. Brundan, Kazhdan–Lusztig polynomials and character formulae for the Lie superalgebra $\mathfrak{q}(n)$, Adv. Math. 182 (2004) 28–77.
[3] J. Brundan, A. Kleshchev, Modular representations of the supergroup $Q(n)$, I, J. Algebra 260 (2003) 64–98.
[4] S.-J. Cheng, W. Wang, Remarks on the Schur–Howe–Sergeev Duality, Lett. Math. Phys. 52 (2000) 143–153.
[5] E. Friedlander, B. Parshall, Modular representation theory of Lie algebras, Amer. J. Math. 110 (1988) 1055–1093.
[6] A. Frisk, V. Mazorchuk, Regular strongly typical blocks of $\mathcal{O}^{\mathfrak{q}}$, preprint, 2008, arXiv:0810.4758.
[7] J. Germoni, Indecomposable representations of special linear Lie superalgebras, J. Algebra 209 (1998) 367–401.
[8] I. Gordon, Representations of semisimple Lie algebras in positive characteristic and quantum groups at roots of unity, Lond. Math. Soc. Lect. Notes Series 290 (2001) 146–167.
[9] R. Holmes, D. Nakano, Brauer-type reciprocity for a class of graded associative algebras, J. Algebra 144 (1991) 117–126.
[10] R. Horn, C. Johnson, Topics in Matrix Analysis, Cambridge University Press, Cambridge, 1994.
[11] J. Jantzen, Representations of Lie algebras in prime characteristic, in: Representation Theories and Algebraic Geometry, Montreal, PQ, 1997, in: NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 514, Kluwer Acad. Publ., Dordrecht, 1998, pp. 185–235.
[12] I. Penkov, Characters of typical irreducible finite-dimensional $\mathfrak{q}(n)$-modules, Funct. Anal. Appl. 20 (1986) 30–37.
[13] A. Premet, Irreducible representations of Lie algebras of reductive groups and the Kac–Weisfeiler conjecture, Invent. Math. 121 (1995) 79–117.
[14] A. Premet, Special transverse slices and their enveloping algebras, Adv. Math. 170 (2002) 1–55.
[15] A. Rudakov, The representations of classical semisimple Lie algebras in characteristic p, Izv. Akad. Nauk SSSR Ser. Mat. 34 (1970) 735–743.
[16] A. Sergeev, The tensor algebra of the identity representation as a module over the Lie superalgebras $\mathfrak{gl}(n, m)$ and $Q(n)$, Math. USSR Sbornik 51 (1985) 419–427.
[17] S. Skryabin, Representations of the Poisson algebra in prime characteristic, Math. Z. 243 (2003) 563–597.
[18] W. Wang, L. Zhao, Representations of Lie superalgebras in prime characteristic I, Proc. Lond. Math. Soc. 99 (2009) 145–167.
[19] B. Weisfeiler, V. Kac, On irreducible representations of Lie p-algebras, Func. Anal. Appl. 5 (1971) 111–117.
[20] J. Xiao, Finite-dimensional representations of $U_t(\mathrm{sl}(2))$ at roots of unity, Canad. J. Math. 49 (1997) 772–787.
[21] L. Zhao, Finite W-superalgebras for queer Lie superalgebras and higher Sergeev duality, preprint, arXiv:1012.2326.